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WAMIT
THEORY MANUAL

by

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Chapter 1

INTRODUCTION

This **Theory Manual** describes the theoretical background of the computer program WAMIT (WaveAnalysisMIT). It provides an overview of the theories and the computational methodologies in a concise manner. For detailed information on the selected subjects, some of the published and unpublished articles are included in the Appendix. The use of the program WAMIT and its capabilities are described in a separate **User Manual**.

WAMIT is a panel program designed to solve the boundary-value problem for the interaction of water-waves with prescribed bodies in finite- and infinite- water depth. Viscous effects of the fluid are not considered throughout and thus the flow field is potential without circulation. A perturbation series solution of the nonlinear boundary value problem is postulated with the assumption that the wave amplitude is small compared to the wave length. It is also assumed that the body stays at its mean position and, if it is not fixed, the oscillatory amplitude of the body motion is of the same order as the wave amplitude. The time harmonic solutions corresponding to the first- and second-terms of the series expansion are solved for a given steady-state incident wave field. The incident wave field is assumed to be represented by a superposition of the fundamental first-order solutions of particular frequency components in the absence of the body. The boundary value problem is recast into integral equations using the wave source potential as a Green function. The integral equation is then solved by a 'panel' method for the unknown velocity potential and/or the source strength on the body surface. Using the latter, the fluid velocity on the body surface is evaluated.

For incident waves of specific frequencies and wave headings, the linear problem is solved first. Chapter 2 describes the linear boundary value problem and the solution procedure. The quadratic interaction of two linear solutions defines the boundary condition for the second-order problem. Chapter 3 is devoted to the discussion of the second-order problem. In Chapter 4, the rigid body motion is discussed in the context of the perturbation series and the expressions for the hydrodynamic forces are derived. Chapter 5 lists the references quoted in this manual and Chapter 6 lists the references illustrating the computational results obtained from WAMIT. The Appendix includes the articles on the subjects which are not explained in detail in this manual.

Since the following will focus primarily on the hydrodynamic interaction, it is appropriate to mention here restrictions on the bodies which may be analyzed using WAMIT. The bodies may be fixed, constrained or neutrally buoyant and they may be bottom mounted, submerged or surface piercing. Multibody interactions between any combinations of these bodies can be analyzed. If only first-order quantities are required, the bodies need not be rigid. Flexible bodies can be analyzed if the oscillatory displacements of the body surface are described by specified mode shapes.

An extremely thin *submerged* body may be difficult to analyze when the small thickness between the two facing surfaces compared to the other dimensions of the body may render the integral equation ill-conditioned[1]. However, a thin body floating on the free surface (in other words the draft is very small) is not included in this category, since it has only one large wetted surface. In the extreme case of the latter, a body touching the free surface may be analyzed.

1.1 Description of the Problem

The flow is assumed to be potential, free of separation or lifting effects, and it is governed by the velocity potential $\Phi(\mathbf{x}, t)$ which satisfies Laplace's equation in the fluid domain:

$$\nabla^2 \Phi = 0 \tag{1.1}$$

Here t denotes the time and $\mathbf{x}$=(x, y, z) denotes the Cartesian coordinates of a point in space. $\mathbf{x}$ may be expressed as the sum of the component vectors as $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The undisturbed free surface is the $z = 0$ plane with the fluid domain $z < 0$.

The fluid velocity is given by the gradient of the velocity potential

$$\mathbf{V}(\mathbf{x}, t) = \nabla\Phi = \frac{\partial\Phi}{\partial x}\mathbf{i} + \frac{\partial\Phi}{\partial y}\mathbf{j} + \frac{\partial\Phi}{\partial z}\mathbf{k} \tag{1.2}$$

The pressure follows from Bernoulli's equation:

$$p(X, t) = -\rho(\frac{\partial\Phi}{\partial t} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi + gz) \tag{1.3}$$

where ρ is the density of the fluid and g is the gravitational acceleration.

The velocity potential satisfies the nonlinear free-surface condition:

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} + 2\nabla\Phi \cdot \nabla\frac{\partial\Phi}{\partial t} + \frac{1}{2}\nabla\Phi \cdot \nabla(\nabla\Phi \cdot \nabla\Phi) = 0 \tag{1.4}$$

applied on the exact free surface

$$\zeta(x, y) = -\frac{1}{g}(\frac{\partial\Phi}{\partial t} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi)_{z=\zeta} \tag{1.5}$$

[1]See Martin and Rizzo (1993)

With the assumption of a perturbation solution in terms of a small wave slope of the incident waves, the velocity potential is expanded in a form

$$\Phi(\mathbf{x},t) = \Phi^{(1)}(\mathbf{x},t) + \Phi^{(2)}(\mathbf{x},t) + \cdots \tag{1.6}$$

When the body is not fixed the motion amplitude[2] of the body is also expanded in a perturbation series

$$\boldsymbol{\xi} = \boldsymbol{\xi}^{(1)} + \boldsymbol{\xi}^{(2)} + \cdots \tag{1.7}$$

From (1.5) and (1.6) and with aid of the Taylor expansion with respect to the mean free surface, the free surface elevation also takes the form of the perturbation series,

$$\zeta(x,y) = \zeta^{(1)}(x,y) + \zeta^{(2)}(x,y) + \cdots \tag{1.8}$$

with

$$\zeta^{(1)}(x,y) = -\frac{1}{g}\frac{\partial \Phi}{\partial t}^{(1)} \tag{1.9}$$

$$\zeta^{(2)}(x,y) = -\frac{1}{g}\left(\frac{\partial \Phi}{\partial t}^{(2)} + \frac{1}{2}\nabla\Phi^{(1)}\cdot\nabla\Phi^{(1)} - \frac{1}{g}\frac{\partial \Phi}{\partial t}^{(1)}\frac{\partial^2\Phi^{(1)}}{\partial z\partial t}\right) \tag{1.10}$$

In the equations (1.9) and (1.10), the right-hand sides are evaluated on $z = 0$.

From the equations (1.4), (1.6) and (1.8-10), the free surface boundary conditions for $\Phi^{(1)}$ and $\Phi^{(2)}$ imposed on $z = 0$ are derived. For a moving body, from the equations (1.6) and (1.7) and with the Taylor expansion of Φ with respect to the mean body surface, the body boundary condition is derived. These boundary conditions are discussed in later chapters.

Given a wave spectrum, it is customary to assume the spectrum is expressed as a linear superposition of the first-order incident waves of different frequencies. Thus the total first order potential for the wave-body interaction can be expressed by a sum of components having circular frequency $\omega_j > 0$:

$$\Phi^{(1)}(\mathbf{x},t) = Re\sum_j \phi_j(\mathbf{x})e^{i\omega_j t} \tag{1.11}$$

Here we introduce the complex velocity potential $\phi_j(\mathbf{x})$, indepedent of time, with the understanding that the real part of the time harmonic solution is physically relevant.

In (1.11) $\phi_j(\mathbf{x})$ denotes the first-order solution in the presence of the incident wave of frequency ω_j and the wave heading β_j. The directional spreading of the incident waves is not shown explicitly in (1.11). However if it is necessary, one can easily incorporate it by expressing $\phi_j(\mathbf{x})$ as a sum of the velocity potential components each of which corresponds to a particular wave heading.

At the second-order, the total velocity potential takes a form

[2]Here $\boldsymbol{\xi}$ includes both the translational and the rotational modes of the rigid body. Frequently $\boldsymbol{\xi}$ is used, in a restrictive sense, to denote only the translational modes. In that case $\boldsymbol{\alpha}$ is used for the rotational modes.

$$\Phi^{(2)}(\mathbf{x},t) = Re \sum_i \sum_j \phi_{ij}^+(\mathbf{x}) e^{i(\omega_i+\omega_j)t} + \phi_{ij}^-(\mathbf{x}) e^{i(\omega_i-\omega_j)t} \quad (1.12)$$

with symmetric conditions

$$\phi_{ij}^+ = \phi_{ji}^+ \quad \text{and} \quad \phi_{ij}^- = \phi_{ji}^{-*} \quad (1.13)$$

Here we may assume $\omega_i \geq \omega_j \geq 0$ without loss of the generality. ϕ_{ij}^+ and ϕ_{ij}^- are referred to as the sum- and difference-frequency velocity potential with frequencies $\omega_i + \omega_j$ and $\omega_i - \omega_j$, respectively. These are determined by two linear incident wave components: one has frequency ω_j and wave heading β_j and the other ω_i and β_i. If one consider the directional spreading of the incident waves, $\phi_{ij}^{\pm}(\mathbf{x})$ are represented by a double summation over wave headings (β_i, β_j).

In the next two chapters, we discuss the solution procedures for the complex velocity potentials ϕ_j and $\phi_{ij}^{\pm}$ in (1.11) and (1.12).

Chapter 2

THE FIRST-ORDER PROBLEM

In this Chapter, we review the first-order boundary value problem and its solution procedure. Here we consider one particular frequency component, ω_j from the discrete spectrum (1.12). Thus we omit the subscript j indicating the frequency component in this Chapter.

The velocity potential of the first-order incident wave is defined by

$$\phi_I = \frac{igA}{\omega} Z(\kappa z) e^{-i\kappa(x\cos\beta + y\sin\beta)} \tag{2.1}$$

representing a plane progressive wave of the circular frequency ω and the wave heading angle β. β is an angle of incidence to the positive $x-$ axis. A denotes the complex wave amplitude.

The function Z represents the depth dependence of the flow and is given by

$$Z(\kappa z) = \mathrm{e}^{\kappa z} \tag{2.2}$$

for infinite water depth, where $\kappa = \dfrac{\omega^2}{g} \equiv \nu$. For a fluid of depth h, it is given by

$$Z(\kappa z) = \frac{\cosh(\kappa(z+h))}{\cosh(\kappa h)} \tag{2.3}$$

where the wavenumber κ is the real root of the dispersion relation

$$\kappa \tanh \kappa h = \frac{\omega^2}{g} \tag{2.4}$$

An efficient root finding algorithm for the computation of κ from equation (2.4) is explained in Newman (1990).

The scattering velocity potential ϕ_S represents the disturbance to the incident wave due to the presence of the body in its fixed position. Linearity of the problem allows it to be distinguished from the disturbance due to the the motion of the body which is represented by the radiation potential ϕ_R. Thus the total velocity potential is given by

$$\phi = \phi_I + \phi_S + \phi_R = \phi_D + \phi_R \tag{2.5}$$

where the diffraction potential ϕ_D is defined to be the sum of ϕ_I and ϕ_S.

The radiation potential itself is a linear combination of the components corresponding to the modes of motion such that

$$\phi_R = i\omega \sum_{k=1}^{6} \xi_k \phi_k \tag{2.6}$$

Here ξ_k is the complex amplitude of the oscillatory motion in mode k of the six degrees of freedom, and ϕ_k the corresponding unit-amplitude radiation potential (specifically, unit-amplitude means the unit-amplitude linear or angular **velocity** of the rigid body motion). These modes are referred to as surge, sway, heave, roll, pitch and yaw in the increasing order of j.

2.1 Boundary Value Problem

The total first-order velocity potential ϕ, along with each of its components, satisfies the Laplace equation in the fluid domain:

$$\nabla^2 \phi = 0, \tag{2.7}$$

the linear free-surface condition:

$$\phi_z - \nu\phi = 0 \quad \text{on} \quad z = 0, \tag{2.8}$$

and a condition on the sea bottom:

$$\nabla\phi \to 0 \quad \text{as} \quad z \to -\infty, \tag{2.9a}$$

or

$$\phi_z = 0 \quad \text{on} \quad z = -h \tag{2.9b}$$

for infinite- and finite- water depth, respectively.

In addition, the scattering and the radiation potentials are subject to a radiation condition stating that the wave energy associated with the disturbance due to the body is carried away from the body in all direction in the far field.

Finally, the conditions on the body surface complete the description of the boundary value problem. They take the form

$$\frac{\partial \phi_k}{\partial n} = n_k \tag{2.10}$$

and

$$\frac{\partial \phi_S}{\partial n} = -\frac{\partial \phi_I}{\partial n} \tag{2.11}$$

where $(n_1, n_2, n_3) = \mathbf{n}$ and $(n_4, n_5, n_6) = \mathbf{x} \times \mathbf{n}$. The unit vector $\mathbf{n}$ is normal to the body boundary and it is assumed the normal vector points out of the fluid domain. $\mathbf{x}$ is the position of a point on the body boundary. From (2.5), it follows that

$$\frac{\partial \phi_D}{\partial n} = 0 \tag{2.12}$$

Variations to the canonical problem specified above include multibody interactions, generalized modes and the presence of the vertical walls.[1] These topics are reviewed briefly next, along with the exploitation of the symmetry of the bodies.

2.1.1 Multiple bodies

The decomposition of the radiation potential into components, corresponding to the modes of the rigid body motion, can be extended to multi-body interaction. This is done by defining ϕ_k as the velocity potential corresponding to a particular mode of one body while the other bodies are kept stationary. In this way, the total radiation potential consists of $6N$ components, where N is the number of bodies.

2.1.2 Body deformations

Appication can also be made to analyze the more general modes of motion of the structure, beyond the rigid body motions. Examples are the bending modes of structural deformation and the compressible modes of a flexible body. To analyze these modes, the mode shapes are described in terms of the vector displacement of the body boundary. Let $\mathbf{u}_k(\mathbf{x})$ be the mode shape given as a function of the points on the body boundary. Then the corresponding radiation potential ϕ_k satisfies

$$\frac{\partial \phi_k}{\partial n} = \mathbf{u}_k \cdot \mathbf{n} \tag{2.13}$$

on the body boundary. The details of the analysis for generalized modes and its applications are described in Newman (1994).

2.1.3 Bodies near walls

When walls are present, the velocity potential is subject to the condition

$$\frac{\partial \phi}{\partial n} = 0 \tag{2.14}$$

on the walls. If we define the incident wave as in (2.1), a reflected wave (ϕ_I^r) should be added to ϕ_I to describe the wave field in the absence of the body. The resultant wave field

[1] Walls are assumed to coincide with the $x = 0$ or the $y = 0$ planes. When there are two walls they meet perpendicularly at the origin.

satisfies (2.14) and it is a standing wave with maximum free surface elevation $2|A|$. (The actual free surface elevation depends on the incident wave heading β.) The total velocity potential is then expressed in the form

$$\phi = \phi_I + \phi_I^r + \phi_S + \phi_R \tag{2.15}$$

If the body geometry is symmetric with respect to the $x = 0$ plane, the $y = 0$ plane or both, the computational domain for ϕ_D, ϕ_S and ϕ_R can be reduced to half or one quadrant of the entire domain. In these cases, the velocity potentials are expressed as a sum of the symmetric and antisymmetric components with respect to the planes of symmetry. On these planes, the symmetric component satisfies

$$\frac{\partial \phi}{\partial n} = 0 \tag{2.16}$$

and the antisymmetric component satisfies

$$\phi = 0 \tag{2.17}$$

2.1.4 Integral Equations

The boundary value problem is solved by the integral equation method. Thus the velocity potential ϕ_k on the body boundary is obtained from the integral equation

$$2\pi\phi_k(\mathbf{x}) + \iint_{S_B} d\boldsymbol{\xi}\phi_k(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} = \iint_{S_B} d\boldsymbol{\xi} n_k G(\boldsymbol{\xi};\mathbf{x}) \tag{2.18}$$

where S_B denotes the body boundary.

The corresponding equation for the total diffraction velocity potential ϕ_D is (Korsmeyer et al (1988))

$$2\pi\phi_D(\mathbf{x}) + \iint_{S_B} d\boldsymbol{\xi}\phi_D(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} = 4\pi\phi_I(\mathbf{x}) \tag{2.19}$$

Alternatively, the scattering potential can be obtained from

$$2\pi\phi_S(\mathbf{x}) + \iint_{S_B} d\boldsymbol{\xi}\phi_S(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} = \iint_{S_B} d\boldsymbol{\xi}(-\frac{\partial \phi_I}{\partial n})G(\boldsymbol{\xi};\mathbf{x}) \tag{2.20}$$

and the diffraction potential follows from (2.5). From a computational point of view, (2.19) is slightly more efficient than (2.20) due to the simpler form of the right-hand-side.

We need to evaluate the fluid velocity on the body surface and in the fluid domain for the computation of the second-order wave forces as discussed in Chapter 4. The velocity may be computed from the spacial derivatives of Green's integral equations, (2.18) to (2.20). However when these equations are solved using a low order panel method, they can not predict the velocity accurately on or close to the body boundary. This is due to the hypersingular integral arising from the double derivative of the constant strength wave source potential distributed on a quadrilateral panel. For this reason, the fluid velocity is computed based on the source formulation. The integral equation for the source strength σ_k corresponding to the radiation potential ϕ_k takes the form (Lee and Newman (1991))

$$2\pi\sigma_k(\mathbf{x}) + \iint_{S_B} d\boldsymbol{\xi}\sigma_k(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\mathbf{x}}} = n_k \tag{2.21}$$

and that of σ_S corresponding to the scattering potential ϕ_S

$$2\pi\sigma_S(\mathbf{x}) + \iint_{S_B} d\boldsymbol{\xi}\sigma_S(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\mathbf{x}}} = -\frac{\partial \phi_I}{\partial n} \tag{2.22}$$

The fluid velocity on the body boundary or in the fluid domain due to ϕ_k or ϕ_S is then obtained from

$$\nabla\phi(\mathbf{x}) = \nabla \iint_{S_B} d\boldsymbol{\xi}\sigma(\boldsymbol{\xi})G(\boldsymbol{\xi};\mathbf{x}) \tag{2.23}$$

The fluid velocity due to the incident wave is evaluated directly from (2.1).

Integral equations, (2.18) to (2.22) are solved by the panel method. The wetted body surface is represented by an ensemble of quadrilateral panels (a triangular panel is a special type of quadrilateral panel where two vertices coalesce). The unknowns are assumed to be constant over each panel and the integral equation is enforced at the centroid of each panel (a collocation method). For example, the discrete form of the equation (2.20) takes the form

$$\begin{aligned} 2\pi\phi_S(\mathbf{x}_k) \quad &+ \sum_{n=1}^{NEQN} \phi_S(\mathbf{x}_n) \int_{S_n} d\boldsymbol{\xi} \frac{\partial G(\boldsymbol{\xi};\mathbf{x}_k)}{\partial n_{\boldsymbol{\xi}}} \\ &= \sum_{n=1}^{NEQN} -\frac{\partial \phi_I(\mathbf{x}_n)}{\partial n} \int_{S_n} d\boldsymbol{\xi} G(\boldsymbol{\xi};\mathbf{x}_k) \end{aligned} \tag{2.24}$$

where $NEQN$ is the total number of panels (unknowns) and $\mathbf{x}_k$ are the coordinates of the centroid of the k-th panel.

2.1.5 The Green Function

The Green function $G(\mathbf{x};\boldsymbol{\xi})$, which is referred to as the wave source potential, is the velocity potential at the point $\mathbf{x}$ due to a point source of strength -4π located at the point $\boldsymbol{\xi}$. It satisfies the free-surface and radiation conditions, and in infinite water depth it is defined by (Wehausen and Laitone (1960))

$$G(\mathbf{x};\boldsymbol{\xi}) = \frac{1}{r} + \frac{1}{r'} + \frac{2\nu}{\pi}\int_0^\infty dk\, \frac{e^{k(z+\zeta)}}{k-\nu} J_0(kR) \tag{2.25}$$

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 \tag{2.26}$$

$$r'^2 = (x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2 \tag{2.27}$$

where $J_0(x)$ is the Bessel function of zero order. In finite depth, it is defined by

$$G(\mathbf{x};\boldsymbol{\xi}) = \frac{1}{r} + \frac{1}{r''} + 2\int_0^\infty dk \; \frac{(k+\nu)\cosh k(z+h)\cosh k(\zeta+h)}{k\sinh kh - \nu\cosh kh} \; e^{-kh} J_0(kR) \tag{2.28}$$

$$(r'')^2 = (x-\xi)^2 + (y-\eta)^2 + (z+\zeta+2h)^2. \tag{2.29}$$

In both expressions (2.25) and (2.28), the Fourier $k-$integration is indented above the pole on the real axis in order to enforce the radiation condition.

In the equation (2.24), the influence due to the continuous distribution of the Rankine part of the wave source potential on a quadrilateral panel is evaluated based on the algorithms described in Newman (1985). The remaining wave part of the Green function is evaluated based on the algorithms described in Newman (1992). The integration of the latter part over a panel is carried out using either one or four point Gauss quadrature. When the field and source points are close together and in the vicinity of the free surface, G takes a form

$$G(\mathbf{x};\boldsymbol{\xi}) = \frac{1}{r} + \frac{1}{r'} - 2\nu \mathrm{e}^{\nu(z+\zeta)}(\log(r' + |z+\zeta|) + (\gamma - \log 2) + r' + O(r'\log r')) \tag{2.30}$$

where γ is the Euler constant. The logarithmic singularity in (2.30) may not be evaluated accurately using Gauss quadrature in this situation. The algorithm for evaluating the influence of the logarithmic singularity distributed over a panel can be found in Newman and Sclavounos (1988). When the panel is on the free surface, special care is required for the evaluation of Green function as is discussed in Newman (1993) and Zhu (1994). The former is attached in the appendix.

When there are walls or when we exploit the symmetry/antisymmetry of the flow, the velocity potentials are subject to the conditions (2.14), (2.16) or (2.17). In this case, the image source must be placed at the reflected points of $\mathbf{x}$ with respect to the planes of symmetry. The Green function is then modified to be a sum of (2.15) or (2.28) and its image sources.

2.1.6 Linear system

The linear system, generated by implementing the solution procedure described above (for example (2.24)), is solved by an iterative method, a block iterative method or Gauss elimination. The details on the iterative method is described in Lee (1988). The relative computational efficiency between the iterative method and Gauss elimination can be illustrated by the ratio of the number of floating point operations. This ratio, the former to the latter, is $O(\frac{3KM}{NEQN})$. Where NEQN is the number of unknowns or the dimension of the linear system and K is the number of iterations. M is the number of the right-hand sides of the linear system sharing a left-hand side. (As an example, in (2.24) the number of the right-hand-side vectors is the same as the number of the different wave headings of ϕ_I for a fixed left-hand-side matrix.) In most applications the iterative method converges in 10-15 iterations and is the most efficient way to solve the linear system. Gauss elimination is not only slower than the iterative method for large $NEQN$ but also requires sufficiently large core memory to store the entire matrix elements, since the LU decomposition is performed using the core memory. On the other hand, in the iterative method, all or part of the matrix may be stored on the auxiliary storage devise (hard disk) and then retrieved at each iteration.

There are some problems for which the iterative method is slowly convergent or non-convergent due to bad conditioning of the linear system. Some examples are i) barges with shallow draft, ii) bodies with large flare and iii) multiple bodies separated by small gaps. Another case where the iterative method may be slowly convergent is in the linear system for the extended boundary integral equation (see Section 2.3). For these problems, Gauss elimination or the block iterative method should be used. The block iterative method is based on the same algorithm as the iterative method, but Gauss elimination is applied locally for the specified diagonal blocks. At each stage of the iterations, the back substitution is performed for the diagonal blocks. This accelerates the rate of convergence as the dimension of the block increases. The limiting case is the same as Gauss elimination. The opposite limit is the case when the dimension of the blocks is one, which is identical to the iterative method. The block iterative method requires core memory to store the elements of one block for LU decomposition.

2.1.7 Removal of Irregular Frequencies

The integral equations (2.18-22) have nonunique solutions at the common irregular frequencies corresponding to the Dirichlet eigen-frequencies for the closed domain defined by the body boundary, and the interior free surface, S_i. The eigenmodes correspond to the solution with the homogeneous Dirichlet condition on S_B and the linear free-surface condition on S_i. Numerical solutions of these equations are erroneous near the irregular frequencies. The extended boundary integral equation method is applied to remove the irregular frequency effect from the velocity potentials in (2.18-20).

The extended boundary integral equations for ϕ_k are

$$2\pi\phi_k(\mathbf{x}) + \iint_{S_B} d\boldsymbol{\xi}\phi_k(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} + \int_{S_f} d\boldsymbol{\xi}\phi'_k(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} = \iint_{S_B} d\boldsymbol{\xi} n_k G(\boldsymbol{\xi};\mathbf{x}) \qquad (2.31a)$$

$$-4\pi\phi'_k(\mathbf{x}) + \iint_{S_B} d\boldsymbol{\xi}\phi_k(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} + \int_{S_f} d\boldsymbol{\xi}\phi'_k(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} = \iint_{S_B} d\boldsymbol{\xi} n_k G(\boldsymbol{\xi};\mathbf{x}) \qquad (2.31b)$$

where ϕ'_k is an artificial velocity potential defined in the interior domain. Equations (2.31a) and (2.31b) are for $\mathbf{x}$ on S_B and S_i, respectively. These two equations are solved simultaneously for ϕ_k on S_B and ϕ'_k on S_i. We discard ϕ'_k after we solve the equations, since only ϕ_k on S_B is physically relevant.

The equations for the diffraction potential ϕ_D are given by

$$2\pi\phi_D(\mathbf{x}) + \iint_{S_B} d\boldsymbol{\xi}\phi_D(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} + \int_{S_f} d\boldsymbol{\xi}\phi'_D(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} = 4\pi\phi_I(\mathbf{x}) \qquad (2.32a)$$

$$-4\pi\phi'_D(\mathbf{x}) + \iint_{S_B} d\boldsymbol{\xi}\phi_D(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} + \int_{S_f} d\boldsymbol{\xi}\phi'_D(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} = 4\pi\phi_I(\mathbf{x}) \qquad (2.32b)$$

The scattering potential takes the same form as the radiation potential but $-\dfrac{\partial \phi_I}{\partial n}$ replaces n_k on the right-hand side of the equations (2.31a) and (2.32b).

The extended boundary integral equation for the source formulation of the radiation or scattering problem takes a form

$$2\pi\sigma(\mathbf{x}) + \iint_{S_B} d\boldsymbol{\xi}\sigma(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} + \int_{S_f} d\boldsymbol{\xi}\sigma'(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} = g(\mathbf{x}) \qquad (2.33a)$$

$$-4\pi\sigma'(\mathbf{x}) + \iint_{S_B} d\boldsymbol{\xi}\sigma(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} + \int_{S_f} d\boldsymbol{\xi}\sigma'(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{\xi};\mathbf{x})}{\partial n_{\boldsymbol{\xi}}} = -V(\mathbf{x}) \qquad (2.33b)$$

where $g(\mathbf{x}) = n_k$ ($g(\mathbf{x}) = -\dfrac{\partial \phi_I}{\partial n}$) for the radition (scattering) problem. The proper condition of the function $V(\mathbf{x})$ is discussed in Lee *et al* (1995).

The details of the derivation of these equations are described in Kleinman (1982), Lee *et al* (1995) and Zhu (1994). Zhu (1994) also makes a comparison with other methods for the removal of the irregular frequency effect and demonstrates the effectiveness of the present method.

Chapter 3

THE SECOND-ORDER PROBLEM

From the quadratic interaction of the two linear wave components of the frequencies ω_i and ω_j in the discrete spectrum (1.11)[1], we have the second-order waves of the frequency components $\omega_i + \omega_j$ and $\omega_i - \omega_j$ in (1.12). In this Chapter we restrict our attention to the solution of the second-order boundary value problem, i.e. the second-order potential. Other second-order quantities, such as the quadratic second-order force , obtainable from the first-order solution, are discussed in Chapter 4.

The second-order velocity potential $\Phi^{(2)}(\mathbf{x}, t)$ is subject to the free surface condition

$$\frac{\partial^2 \Phi^{(2)}}{\partial t^2} + g\frac{\partial \Phi^{(2)}}{\partial z} = Q_F(x, y; t) \tag{3.1}$$

and the body boundary condition

$$\frac{\partial \Phi^{(2)}}{\partial n} = Q_B(\mathbf{x}; t) \tag{3.2}$$

where the inhomogeneous right-hand-side of the second-order free-surface condition (3.1) defines the quadratic forcing function (Newman (1977, Section 6.4))

$$Q_F = \frac{1}{g}\frac{\partial \Phi^{(1)}}{\partial t}\frac{\partial}{\partial z}(\frac{\partial^2 \Phi^{(1)}}{\partial t^2} + g\frac{\partial \Phi^{(1)}}{\partial z}) - \frac{\partial}{\partial t}(\nabla\Phi^{(1)} \cdot \nabla\Phi^{(1)}) \tag{3.3}$$

Q_F is to be evaluated on $z = 0$. The forcing function on the body boundary is given by (Ogilvie (1983))

$$Q_B = -\frac{\partial \Phi_I^{(2)}}{\partial n} + \mathbf{n} \cdot \frac{\partial H}{\partial t}\mathbf{x} + (\boldsymbol{\alpha}^{(1)} \times \mathbf{n}) \cdot (\frac{\partial(\boldsymbol{\xi}^{(1)} + \boldsymbol{\alpha}^{(1)} \times \mathbf{x})}{\partial t} - \nabla\Phi^{(1)})$$
$$-\mathbf{n} \cdot ((\boldsymbol{\xi}^{(1)} + \boldsymbol{\alpha}^{(1)} \times \mathbf{x}) \cdot \nabla)\nabla\Phi^{(1)} + \sum_{k=1}^{6}\frac{\partial \xi_k^{(2)}}{\partial t}n_k \tag{3.4}$$

[1]When body is not fixed, the first-order motion affects the second-order solution as shown in the equation (3.4) below.

and it is to be evaluated on the mean body boundary. H in (3.4) is the quadratic components of the coordinate transformation matrix due to the rotation of the body and it takes a form (3.14) and (3.15), for sum- and differency-frequency problems. In (3.4), $\Phi_I^{(2)}$ is referred to as the second-order incident wave potential. This is the second-order potential in the absence of the body and will be discussed in Section 3.2. n_k is the appropriate component of $\mathbf{n}$ for the translational modes and of a component of $(\mathbf{x}\times\mathbf{n})$ for the rotational modes.

In the following evaluations of second-order products of two first-order oscillatory quantities, use is made of the relation

$$\mathrm{Re}(A\mathrm{e}^{i\omega_i t})\mathrm{Re}(B\mathrm{e}^{i\omega_j t}) = \frac{1}{2}\mathrm{Re}[(A\mathrm{e}^{i\omega_i t})(B\mathrm{e}^{i\omega_j t} + B^*\mathrm{e}^{-i\omega_j t})] \tag{3.5}$$

where $(^*)$ denotes the complex conjugate.

In accordance with the definition of the second-order potential (1.12), Q is expressed as

$$Q(\mathbf{x}, t) = \mathrm{Re}\sum_i \sum_j \left[Q_{ij}^+(\mathbf{x})e^{i(\omega_i+\omega_j)t} + Q_{ij}^-(\mathbf{x})e^{i(\omega_i-\omega_j)t}\right] \tag{3.6}$$

With the symmetry condition,

$$Q_{ij}^+ = Q_{ji}^+ \quad \text{and} \quad Q_{ij}^- = Q_{ji}^{-*} \tag{3.7}$$

The other second-order quantities such as the motion amplitude $\boldsymbol{\xi}^{(2)} = \xi_{ij}^{\pm}$ are expressed in a form of (3.6) with the symmetry condition (3.7). With this in mind, we will omit the subscript ij hereafter.

From combining (1.11) and (3.3), we have the expressions for the complex amplitudes of the free-surface forcing functions. Sum- and difference- frequency forcings on $z = 0$ are given by

$$\begin{aligned} Q_F^+ = \frac{i}{4g}\omega_i\phi_i(-\omega_j^2\frac{\partial\phi_j}{\partial z} + g\frac{\partial^2\phi_j}{\partial z^2}) + \frac{i}{4g}\omega_j\phi_j(-\omega_i^2\frac{\partial\phi_i}{\partial z} + g\frac{\partial^2\phi_i}{\partial z^2}) \\ -\frac{1}{2}i(\omega_i+\omega_j)\nabla\phi_i\cdot\nabla\phi_j \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} Q_F^- = \frac{i}{4g}\omega_i\phi_i(-\omega_j^2\frac{\partial\phi_j^*}{\partial z} + g\frac{\partial^2\phi_j^*}{\partial z^2}) - \frac{i}{4g}\omega_j\phi_j^*(-\omega_i^2\frac{\partial\phi_i}{\partial z} + g\frac{\partial^2\phi_i}{\partial z^2}) \\ -\frac{1}{2}i(\omega_i-\omega_j)\nabla\phi_i\cdot\nabla\phi_j^* \end{aligned} \tag{3.9}$$

For future reference, we note that the first-order potential ϕ_i and ϕ_j in (3.8-9), consist of the incident wave potential (ϕ_I) and the body disturbances ($\phi_B = \phi_S + \phi_R$). Thus we may decompose $Q_F^{\pm}$ into the quadratic interactions of the two incident wave potentials

($Q_{II}^{\pm}$), the incident and the body disturbance waves ($Q_{IB}^{\pm}$), and two body disturbance waves ($Q_{BB}^{\pm}$) such that

$$Q_F^{\pm} = Q_{II}^{\pm} + Q_{IB}^{\pm} + Q_{BB}^{\pm} \tag{3.10}$$

Next we consider the sum- and difference-frequency forcing on the body boundary. They are given by

$$\begin{aligned} Q_B^{+} = & - \frac{\partial \phi_I^{+}}{\partial n} + \frac{i(\omega_i + \omega_j)}{2} \mathbf{n} \cdot H^{+} \mathbf{x} \\ & + \frac{1}{4}[(\boldsymbol{\alpha}_i \times \mathbf{n}) \cdot (i\omega_j(\boldsymbol{\xi}_j + \boldsymbol{\alpha}_j \times \mathbf{x}) - \nabla\phi_j) + (\boldsymbol{\alpha}_j \times \mathbf{n}) \cdot (i\omega_i(\boldsymbol{\xi}_i + \boldsymbol{\alpha}_i \times \mathbf{x}) - \nabla\phi_i)] \\ & - \frac{1}{4}\mathbf{n} \cdot [((\boldsymbol{\xi}_i + \boldsymbol{\alpha}_i \times \mathbf{x}) \cdot \nabla)\nabla\phi_j + ((\boldsymbol{\xi}_j + \boldsymbol{\alpha}_j \times \mathbf{x}) \cdot \nabla)\nabla\phi_i] \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} Q_B^{-} = & - \frac{\partial \phi_I^{-}}{\partial n} + \frac{i(\omega_i - \omega_j)}{2} \mathbf{n} \cdot H^{-} \mathbf{x} \\ & + \frac{1}{4}[(\boldsymbol{\alpha}_i \times \mathbf{n}) \cdot (-i\omega_j(\boldsymbol{\xi}_j^* + \boldsymbol{\alpha}_j^* \times \mathbf{x}) - \nabla\phi_j^*) + (\boldsymbol{\alpha}_j^* \times \mathbf{n}) \cdot (i\omega_i(\boldsymbol{\xi}_i + \boldsymbol{\alpha}_i \times \mathbf{x}) - \nabla\phi_i)] \\ & - \frac{1}{4}\mathbf{n} \cdot [((\boldsymbol{\xi}_i + \boldsymbol{\alpha}_i \times \mathbf{x}) \cdot \nabla)\nabla\phi_j^* + ((\boldsymbol{\xi}_j^* + \boldsymbol{\alpha}_j^* \times \mathbf{x}) \cdot \nabla)\nabla\phi_i] \end{aligned} \tag{3.12}$$

The sum- and difference-frequency components of the last term of (3.4),

$$Q_B^{\pm} = \mathrm{i}(\omega_i \pm \omega_j) \sum_k \xi^{\pm} n_k \tag{3.13}$$

are omitted from (3.11-12), since they are not a quadratic function of the first-order solution. These are proportional to the second-order motion and can be treated separately from the rest of body forcing as is discussed below.

The matrices $H^{\pm}$ which account for the rotational motion of the body are given by

$$H^{+} = \frac{1}{2} \begin{pmatrix} -(\alpha_{2i}\alpha_{2j} + \alpha_{3i}\alpha_{3j}) & 0 & 0 \\ \alpha_{1i}\alpha_{2j} + \alpha_{1j}\alpha_{2i} & -(\alpha_{1i}\alpha_{1j} + \alpha_{3i}\alpha_{3j}) & 0 \\ \alpha_{1i}\alpha_{3j} + \alpha_{1j}\alpha_{3i} & \alpha_{2i}\alpha_{3j} + \alpha_{2j}\alpha_{3i} & -(\alpha_{1i}\alpha_{1j} + \alpha_{2i}\alpha_{2j}) \end{pmatrix} \tag{3.14}$$

$$H^{-} = \frac{1}{2} \begin{pmatrix} -(\alpha_{2i}\alpha_{2j}^* + \alpha_{3i}\alpha_{3j}^*) & 0 & 0 \\ \alpha_{1i}\alpha_{2j}^* + \alpha_{1j}^*\alpha_{2i} & -(\alpha_{1i}\alpha_{1j}^* + \alpha_{3i}\alpha_{3j}^*) & 0 \\ \alpha_{1i}\alpha_{3j}^* + \alpha_{1j}^*\alpha_{3i} & \alpha_{2i}\alpha_{3j}^* + \alpha_{2j}^*\alpha_{3i} & -(\alpha_{1i}\alpha_{1j}^* + \alpha_{2i}\alpha_{2j}^*) \end{pmatrix} \tag{3.15}$$

As in the first-order problem, it is convenient to decompose the total second-order potential into three components: the second order incident wave potential ($\phi_I^{\pm}$), the second-order scattering wave potential ($\phi_S^{\pm}$) and the second-order radiation potential ($\phi_R^{\pm}$). We define $\phi_I^{\pm}$ as the potential that satisfies the second-order free surface condition in absence of the body. $\phi_R^{\pm}$ describes the disturbance due the second-order motion of the body and is linearly proportional to the motion amplitude. It can be decomposed into the mode dependent components

$$\phi_R^{\pm} = \mathrm{i}(\omega_i \pm \omega_j) \sum_{k=1}^{6} \xi_k \phi_k \tag{3.16}$$

where the unit-amplitude radiation velocity potential ϕ_k is defined in the same way as ϕ_k in Chapter 2. The rest of the second-order potential is defined to be $\phi_S^{\pm}$, whether the body is fixed or moving. This decomposition is in accordance with the convention for the first-order problem where the wave exciting force is the pressure force due to the sum of the incident and scattering waves.

3.1 Boundary Value Problem

The total second-order potential $\phi^{\pm}$ satisfies Laplace's equation in $z < 0$ and the condition on the sea bottom (equations (2.7) and (2.9a-9b)). The boundary condition on the free surface, on the body boundary and at the far field are specified below for each component potential.

The incident wave potential is subject to

$$-(\omega_i \pm \omega_j)^2 \phi_I^{\pm} + g\frac{\partial \phi_I^{\pm}}{\partial z} = Q_{II}^{\pm} \quad \text{on} \quad z = 0 \tag{3.17}$$

The unit amplitude radiation potential is subject to

$$-(\omega_i \pm \omega_j)^2 \phi_k + g\frac{\partial \phi_k}{\partial z} = 0 \quad \text{on} \quad z = 0 \tag{3.18}$$

and

$$\frac{\partial \phi_k}{\partial n} = n_k \tag{3.19}$$

on the mean body boundary S_B.

Finally, the scattering potential is subject to

$$-(\omega_i \pm \omega_j)^2 \phi_S^{\pm} + g\frac{\partial \phi_S^{\pm}}{\partial z} = Q_{IB}^{\pm} + Q_{BB}^{\pm} \quad \text{on} \quad z = 0 \tag{3.20}$$

and

$$\frac{\partial \phi_S^{\pm}}{\partial n} = Q_B^{\pm} \tag{3.21}$$

At the far field $\phi_R^{\pm}$ satisfies the same radiation condition as that for ϕ_R of the first-order. For $\phi_S^{\pm}$, we apply the 'weak radiation condition' suggested by Molin (1979). Thus based on the asymptotic behaviours of G and $Q_F^{\pm}$ (in other words, from the behaviours of ϕ_i and ϕ_j), the surface integral at the far field can be shown to vanish with the result shown in (3.28). The detailed analysis is in Molin (1979).

3.2 The Second-order Incident Waves

Each of the two first-order incident waves is defined by the amplitude A, frequency ω, and vector wavenumber $\mathbf{K}$ with Cartesian components $(\kappa\cos\beta, \kappa\sin\beta, 0)$. Here β is the angle of incidence relative to the $x-$axis. The first-order incident wave potential (2.1) can be written in the form

$$\phi_I = \frac{\mathrm{i}gA}{\omega} Z(\kappa z) e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{x}} \tag{3.22}$$

On $z = 0$, the first- and the second-derivatives of Z in vertical direction are

$$\Big[\frac{\partial Z(\kappa z)}{\partial z}\Big]_{z=0} = \nu \quad \text{and} \quad \Big[\frac{\partial^2 Z(\kappa z)}{\partial z^2}\Big]_{z=0} = \kappa^2 \tag{3.23}$$

Combining (3.22) with (3.8-9) gives

$$\begin{aligned} Q_{II}^{+} &= -\frac{1}{2} i g^2 A_i A_j \exp(-i(\mathbf{K}_i + \mathbf{K}_j)\cdot\mathbf{x}) \\ &\qquad \Big[\Big(\frac{\kappa_j^2 - \nu_j^2}{2\omega_j}\Big) + \Big(\frac{\kappa_i^2 - \nu_i^2}{2\omega_i}\Big) + \frac{(\omega_i + \omega_j)}{\omega_i\omega_j}(\mathbf{K}_i\cdot\mathbf{K}_j - \nu_i\nu_j)\Big] \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} Q_{II}^{-} &= \frac{1}{2} i g^2 A_i A_j^* \exp(-i(\mathbf{K}_i - \mathbf{K}_j)\cdot\mathbf{x}) \\ &\qquad \Big[\Big(\frac{\kappa_j^2 - \nu_j^2}{2\omega_j}\Big) - \Big(\frac{\kappa_i^2 - \nu_i^2}{2\omega_i}\Big) - \frac{(\omega_i - \omega_j)}{\omega_i\omega_j}\Big(\mathbf{K}_i\cdot\mathbf{K}_j + \nu_i\nu_j\Big)\Big] \end{aligned} \tag{3.25}$$

Solutions for the second-order incident-wave potential follow from (3.17) in the form

$$\phi_I^{\pm} = \frac{Q_{II}^{\pm}(x,y) Z(\kappa_{ij}^{\pm} z)}{-(\omega_i \pm \omega_j)^2 + g\kappa_{ij}^{\pm}\tanh\kappa_{ij}^{\pm} h} \tag{3.26}$$

where

$$\kappa_{ij}^{\pm} = |\mathbf{K}_i \pm \mathbf{K}_j| \tag{3.27}$$

Note that in infinite water depth, $Q_{II} = 0$ if $\beta_i = \beta_j$. Thus the sum frequency incident wave potential ϕ_I^{+} vanishes in unidirectional regular and irregular waves.

3.3 The Second-order Scattering Waves

The solution for the second-order scattering potential, $\phi_S^{\pm}$, is obtained from Green's integral equation

$$
\begin{aligned}
2\pi\phi_S^{\pm}(\mathbf{x}) + \iint_{S_B} \phi_S^{\pm}(\boldsymbol{\xi}) \frac{\partial G(\mathbf{x};\boldsymbol{\xi})}{\partial n_\xi} dS &= \iint_{S_B} Q_B^{\pm}(\boldsymbol{\xi}) G(\mathbf{x};\boldsymbol{\xi}) dS \\
&\quad + \frac{1}{g} \iint_{S_F} Q_F^{\pm}(\boldsymbol{\xi}) G(\mathbf{x};\boldsymbol{\xi}) dS \qquad (3.28)
\end{aligned}
$$

where G is the wave source potential defined in (2.25) and (2.28).

After the solution of (3.28), $\phi_S^{\pm}$ on the free surface and in the fluid domain can be obtained from

$$
\begin{aligned}
4\pi\phi_S^{\pm}(\mathbf{x}) + \iint_{S_B} \phi_S^{\pm}(\boldsymbol{\xi}) \frac{\partial G(\mathbf{x};\boldsymbol{\xi})}{\partial n_\xi} dS &= \iint_{S_B} Q_B^{\pm}(\boldsymbol{\xi}) G(\mathbf{x};\boldsymbol{\xi}) dS \\
&\quad + \frac{1}{g} \iint_{S_F} Q_F^{\pm}(\boldsymbol{\xi}) G(\mathbf{x};\boldsymbol{\xi}) dS \qquad (3.29)
\end{aligned}
$$

The integral equation (3.28) is solved by the panel method. The left-hand side of (3.28) is identical to the integral equation for the first-order potential and thus the discrete form of it is the same as that of (2.24) (but with G corresponding to the same frequency ω and wavenumber as $\phi_S^{\pm}$).

The evaluation of the integrals on the right hand side is described below. We consider the integral over S_B first. For a fixed body, the evaluation $Q_B^{\pm}(\mathbf{x})$ is simple, since it contains only the normal velocity due to $\phi_I^{\pm}$ which are derived in Section 3.2. When the body is not fixed, we first modify the terms involving the double spacial derivative of the first-order velocity potential by applying Stokes's theorem to avoid the inaccurate numerical evaluation of the second-order derivatives based on the lower order panel method (Lee and Zhu (1993)).

$$
\begin{aligned}
&\iint_{S_B} dSG\{\mathbf{n}\cdot[(\boldsymbol{\xi}+\boldsymbol{\alpha}\times\mathbf{x})\cdot\nabla]\nabla\phi\} = \iint_{S_B} dS[\mathbf{n}\cdot(\boldsymbol{\xi}+\boldsymbol{\alpha}\times\mathbf{x})](\nabla\phi\cdot\nabla G) \\
+ &\iint_{S_B} dSG\{\mathbf{n}\cdot[(\nabla\phi\cdot\nabla)(\boldsymbol{\xi}+\boldsymbol{\alpha}\times\mathbf{x})]\} - \iint_{S_B} dS\frac{\partial\phi}{\partial n}[(\boldsymbol{\xi}+\boldsymbol{\alpha}\times\mathbf{x})\cdot\nabla G] \\
+ &\int_{WL} d\mathbf{l}\cdot G[\nabla\phi\times(\boldsymbol{\xi}+\boldsymbol{\alpha}\times\mathbf{x})] \qquad (3.30)
\end{aligned}
$$

After substituting (3.30) into $Q_B^{\pm}$, the integral over S_B is represented by a sum of the integral over each panel assuming $Q_B^{\pm}(\mathbf{x})$ is constant on each panel. $Q_B^{\pm}(\mathbf{x})$ is evaluated on the centroids of the panels. The waterline is approximated by line segments (consisting of the sides of the panels adjacent to the free surface) and $Q_B^{\pm}(\mathbf{x})$ is evaluated on the midpoints of the segments.

The free-surface integral which is displayed as the last term in (3.28) will be considered next in which the 'quadratic forcing function' $Q_F^{\pm}$ is given by (3.8-9). For the purposes of numerical evaluation, the free-surface integral is evaluated separately in two domain divided by a partition circle of radius $\rho = b$. b issufficiently large to neglect the effect of the evanescent waves outside the circle. Here $\rho = \sqrt{x^2+y^2}$. Following the methodology of

Kim and Yue(1989), the integration in the inner domain $\rho < b$ is carried out numerically. In the outer domain ($b \leq \rho$) both the Green function and the asymptotics of the first-order potentials are expanded in Fourier-Bessel series. After integrating the trigonometric functions with respect to the angular coordinate, the free-surface integrals are reduced to the sum of the line integrals with respect to the radial coordinate ρ. The method of integration on the inner domain is described below and that on the outer domain follows in the next Section.

In the inner domain, in order to avoid the evaluation of second-order derivatives of the first-order potentials in (3.8-9), the surface integral is transformed, using Gauss theorem, to a form involving only the first-order derivatives and line integrals around the waterline (WL) and the partition circle (PC) as is shown below.

$$\iint_{S_F} \phi_i \frac{\partial^2 \phi_j}{\partial z^2} G dS = - \iint_{WL+PC} \phi_i (\nabla \phi_j \cdot \mathbf{n}) G dl + \iint_{S_F} [(\nabla \phi_i \cdot \nabla \phi_j) G + \phi_i (\nabla \phi_j \cdot \nabla G)] dS \tag{3.31}$$

where S_F denotes only the inner domain of the free surface. The divergence ∇ and the normal vector $\mathbf{n}$ must be interpreted in the two dimensional sense on the $z = 0$ plane.

The line integrals over WL and PC are carried out in a similar way to the line integral in (3.30). To achieve a better computational efficiency for the remaining surface integral, we divide the inner domain further into two parts separated by a circle of radius 'a' with $a < b$. a is sufficiently large to enclose the body. Inside this circle, the free surface is discretized with quadrilateral panels, in an analogous manner to the body surface. Thus the integration on this region is the sum of the integral over each panels assuming the constant $Q_F^{\pm}$ on each panel evaluated at the centroid. The domain outside the circle of radius a to the partition circle forms an annulus. The integral on this domain is carried out by Gauss-Chebyshev quadrature in the azimuthal direction and Gauss-Legendre quadrature in the radial direction as discussed in Lee and Zhu (1993).

When the field point $\mathbf{x}$ on the body and the source point $\boldsymbol{\xi}$ on the annulus are close, the Rankine singularity of $G(\mathbf{x}; \boldsymbol{\xi})$ renders Gauss-Chebyshev quadrature inefficient. Discussion on the selection of the optimum a is provided in Chapter 11 of the WAMIT User Manual (1995). The analysis on the integral over the far-field free surface is described in detail in Newman (1991) which is included in the Appendix

Chapter 4

THE FIRST- AND SECOND-ORDER FORCES

The expressions for the first- and second-order forces are derived from direct integration of the fluid pressure over a body boundary in Section 4.1. By making use of Green's theorem, part of the forces can be obtained without solving the scattering potential. The first-order Haskind exciting force and the second-order force via indirect approach are discussed in Section 4.2. The equations of motion for the first- and second-order problem are derived in Section 4.3.

4.1 Hydro-static and -dynamic Force and Moment

4.1.1 Coordinate system

We consider three coordinate systems. $\mathbf{X} = (X, Y, Z)$ is a global coordinate system and with $Z = 0$ the undisturbed free-surface. The positive Z axis points upward. $\mathbf{x} = (x, y, z)$ is a body-fixed coordinate system and z also is positive upward when the body is at rest. We introduce a third coordinate system which is fixed in space and coincides with "$\mathbf{x}$ at rest". This coordinate system is denoted by $\hat{\mathbf{x}} = (\hat{x}, \hat{y}, \hat{z})$. Note that $\mathbf{X} = (X, Y, Z)$ and $\hat{\mathbf{x}} = (\hat{x}, \hat{y}, \hat{z})$ are inertial reference frames but $\mathbf{x} = (x, y, z)$ is not. The origin of $\hat{\mathbf{x}}$ may be displaced from the free-surface and Z_o denotes the Z-coordinate of the origin of $\hat{\mathbf{x}}$.

4.1.2 Coordinate transform

The position vectors in the $\hat{\mathbf{x}}-$ and $\mathbf{x}-$ coordinate systems are related to each other by linear transformation

$$\hat{\mathbf{x}} = \boldsymbol{\xi} + T^t \mathbf{x} \tag{4.1}$$

and the normal vector by

$$\hat{\mathbf{n}} = T^t \mathbf{n} \tag{4.2}$$

In (4.1-2), T^t is the transpose of $T = T_3 T_2 T_1$ and T_1, T_2 and T_3 take forms

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix}$$
$$T_2 = \begin{pmatrix} \cos\alpha_2 & 0 & -\sin\alpha_2 \\ 0 & 1 & 0 \\ \sin\alpha_2 & 0 & \cos\alpha_2 \end{pmatrix} \tag{4.3}$$
$$T_3 = \begin{pmatrix} \cos\alpha_3 & \sin\alpha_3 & 0 \\ -\sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ are the translational and rotational displacements of $\mathbf{x}$(-coordinate system) with respect to $\hat{\mathbf{x}}$, respectively. They also represent the motion amplitudes of the body in the order of surge-sway-heave and roll-pitch-yaw. Further discussion on T can be found in Ogilvie (1983).

4.1.3 Pressure integration

In the following derivation, it is understood that the pressure, the velocity potential and the body motion amplitudes are functions of time, though the time t does not appear explicitly.

The total pressure at $\hat{\mathbf{x}}$ is given by Bernoulli's equation

$$\mathbf{P}(\hat{\mathbf{x}}) = -\rho[\Phi_t(\hat{\mathbf{x}}) + \frac{1}{2}\nabla\Phi(\hat{\mathbf{x}}) \cdot \nabla\Phi(\hat{\mathbf{x}}) + g(\hat{z} + Z_o)] \tag{4.4}$$

The hydro-static and -dynamic force and moment are obtained from the integration of the pressure over the instantaneous wetted surface.

$$\mathbf{F} = \iint_{\hat{S}_B} P(\hat{\mathbf{x}})\hat{\mathbf{n}} dS \tag{4.5}$$

and

$$\mathbf{M} = \iint_{\hat{S}_B} P(\hat{\mathbf{x}})(\hat{\mathbf{x}} \times \hat{\mathbf{n}}) dS \tag{4.6}$$

where $\hat{S}_B$ denotes the instantaneous wetted body boundary.

The pressure on the exact body surface ($\hat{\mathbf{x}} \in \hat{S}_B$) may be approximated by Taylor expansion with respect to the mean body surface ($\mathbf{x} \in S_B$)

$$\mathbf{P}(\hat{\mathbf{x}}) = P(\mathbf{x}) + [\boldsymbol{\xi} + (T^t - I)\mathbf{x}] \cdot \nabla P(\mathbf{x}) + ... \tag{4.7}$$

Substituting (4.7) into (4.1-2), we have

$$\hat{\mathbf{x}} = \mathbf{x} + \boldsymbol{\xi}^{(1)} + \boldsymbol{\alpha}^{(1)} \times \mathbf{x} + H\mathbf{x} + \boldsymbol{\xi}^{(2)} + \boldsymbol{\alpha}^{(2)} \times \mathbf{x} + O(A^3) \tag{4.8}$$

$$\hat{\mathbf{n}} = \mathbf{n} + \boldsymbol{\alpha}^{(1)} \times \mathbf{n} + H\mathbf{n} + \boldsymbol{\alpha}^{(2)} \times \mathbf{n} + O(A^3) \tag{4.9}$$

The cross product of (4.8) and (4.9) then takes a form

$$\begin{aligned} \hat{\mathbf{x}} \times \hat{\mathbf{n}} &= \mathbf{x} \times \mathbf{n} + \boldsymbol{\xi}^{(1)} \times \mathbf{n} + \boldsymbol{\alpha}^{(1)} \times (\mathbf{x} \times \mathbf{n}) + \boldsymbol{\xi}^{(1)} \times (\boldsymbol{\alpha}^{(1)} \times \mathbf{n}) \\ &+ H(\mathbf{x} \times \mathbf{n}) + \boldsymbol{\xi}^{(2)} \times \mathbf{n} + \boldsymbol{\alpha}^{(2)} \times (\mathbf{x} \times \mathbf{n}) + O(A^3) \end{aligned} \tag{4.10}$$

From (4-4), (4-7) and (4.11), the pressure is expressed by the values on the mean body position

$$\begin{aligned} \mathbf{P}(\hat{\mathbf{x}}) &= -\rho\{g(z+Z_o) + [\Phi_t^{(1)}(\mathbf{x}) + g(\xi_3^{(1)} + \alpha_1^{(1)}y - \alpha_2^{(1)}x)] \\ &+ [\frac{1}{2}\nabla\Phi^{(1)}(\mathbf{x}) \cdot \nabla\Phi^{(1)}(\mathbf{x}) + (\boldsymbol{\xi}^{(1)} + \boldsymbol{\alpha}^{(1)} \times \mathbf{x}) \cdot \nabla\Phi_t^{(1)}(\mathbf{x}) + gH\mathbf{x} \cdot \nabla z] \\ &+ [\Phi_t^{(2)}(\mathbf{x}) + g(\xi_3^{(2)} + \alpha_1^{(2)}y - \alpha_2^{(2)}x)]\} + O(A^3) \end{aligned} \tag{4.11}$$

H in (4.7-11) is the second-order component of T^t and it takes a form

$$H = \begin{pmatrix} -\frac{1}{2}((\alpha_2^{(1)})^2 + (\alpha_3^{(1)})^2) & 0 & 0 \\ \alpha_1^{(1)}\alpha_2^{(1)} & -\frac{1}{2}((\alpha_1^{(1)})^2 + (\alpha_3^{(1)})^2) & 0 \\ \alpha_1^{(1)}\alpha_3^{(1)} & \alpha_2^{(1)}\alpha_3^{(1)} & -\frac{1}{2}((\alpha_1^{(1)})^2 + (\alpha_2^{(1)})^2) \end{pmatrix} \tag{4.12}$$

Substituting the expansions (4.7-11) into (4.5-6), we have the series expansion of the integrals (4.5-6).

4.1.4 Hydrostatic force and moment of O(1)

These are the buoyancy force and moment when the body is at rest and are expressed by

$$\begin{aligned} \mathbf{F} &= -\rho g \iint_{S_B} (z+Z_o)\mathbf{n}dS \\ \mathbf{M} &= -\rho g \iint_{S_B} (z+Z_o)(\mathbf{x} \times \mathbf{n})dS = -\rho g \iint_{S_B} [(z+Z_o)\mathbf{x}] \times \mathbf{n}dS \end{aligned} \tag{4.13}$$

The following relations are invoked frequently in this Chapter.

$$\begin{aligned} -\iint_S \psi\mathbf{n}dS &= \iiint_V \nabla\psi dV \\ -\iint_S \mathbf{n} \times \psi dS &= \iiint_V \nabla \times \psi dV \\ \nabla \times (z+Z_o)\mathbf{x} &= k \times \mathbf{x} \end{aligned} \tag{4.14}$$

where S is a closed surface consisting of S_B and the waterplane area A_{wp}. V denotes the volume of the body.

Using the relations (4.14), the force and moment are expressed in familiar forms

$$\begin{aligned} \mathbf{F} &= \rho g V k \\ \mathbf{M} &= \rho g V(y_b i - x_b j) \end{aligned} \tag{4.15}$$

where x_b and y_b are the x and y coordinates of the center of buoyancy.

In (4.15), $i, j,$ and k are the unit vectors in $\hat{\mathbf{x}}$ coordinate system.

4.1.5 Linear force and moment

The linear force and moment are obtained from

$$\begin{aligned} \mathbf{F}^{(1)} = &- \rho \iint_{S_B} \mathbf{n}\Phi_t^{(1)} dS \\ &- \rho g \iint_{S_B} (\boldsymbol{\alpha}^{(1)} \times \mathbf{n})(z + Z_o) dS \\ &- \rho \iint_{S_B} \mathbf{n} g(\xi_3^{(1)} + \alpha_1^{(1)} y - \alpha_2^{(1)} x) dS \\ \mathbf{M}^{(1)} = &- \rho \iint_{S_B} (\mathbf{x} \times \mathbf{n})\Phi_t^{(1)} dS \\ &- \rho g \iint_{S_B} (\mathbf{x} \times \mathbf{n})(\xi_3^{(1)} + \alpha_1^{(1)} y - \alpha_2^{(1)} x) dS \\ &- \rho g \iint_{S_B} (\boldsymbol{\xi}^{(1)} \times \mathbf{n})(z + Z_o) dS \\ &- \rho g \iint_{S_B} [\boldsymbol{\alpha}^{(1)} \times (\mathbf{x} \times \mathbf{n})](z + Z_o) dS \end{aligned} \tag{4.16}$$

In (4.16), the first terms are the hydrodynamic force and moment and the rest are the hydrostatic. Following the decomposition (2.5), we consider component potentials such that $\Phi^{(1)} = \Phi_I^{(1)} + \Phi_S^{(1)} + \Phi_R^{(1)} = \Phi_D^{(1)} + \Phi_R^{(1)}$. The hydrodynamic force and moment are divided into two components: the "wave exciting force" due to $\Phi_D^{(1)}$ and the force due to $\Phi_R^{(1)}$ expressed in terms of the added mass and damping coefficients.

The integrals of the hydrostatic pressure can be simplified by applying (4.14) and their variations with the results

$$\mathbf{F}^{(1)} = - \rho \iint_{S_B} \Phi_t^{(1)} \mathbf{n} dS - \rho g A_{wp}(\xi_3^{(1)} + \alpha_1^{(1)} y_f - \alpha_2^{(1)} x_f) k$$

$$
\begin{aligned}
\mathbf{M}^{(1)} = & - \rho \iint_{S_B} (\mathbf{x} \times \mathbf{n}) \Phi_t^{(1)} dS \\
& - \rho g[-V\xi_2^{(1)} + A_{wp} y_f \xi_3^{(1)} + (Vz_b + L_{22})\alpha_1^{(1)} - L_{12}\alpha_2^{(1)} - Vx_b\alpha_3^{(1)}]i \\
& - \rho g[\ \ V\xi_1^{(1)} - A_{wp} x_f \xi_3^{(1)} - L_{12}\alpha_1^{(1)} + (Vz_b + L_{11})\alpha_2^{(1)} - Vy_b\alpha_3^{(1)}]j
\end{aligned}
\tag{4.17}
$$

where L_{ij} is the second moment over the waterplane area. For example, $L_{12} = \iint_{A_{wp}} xy dS$. x_f and y_f are the coordinates of the center of floatation.

Following (1.10), the force can be represented by a discrete spectrum (the moment takes an identical form and is omitted here)

$$
\mathbf{F}^{(1)} = \mathrm{Re} \sum_j \mathbf{F}_j^{(1)} \mathrm{e}^{\mathrm{i}\omega_j t} \tag{4.18}
$$

4.1.6 Second-order force and moment

The second-order force and moment are obtained from

$$
\begin{aligned}
\mathbf{F}^{(2)} = & - \rho g \iint_{S_B} (z + Z_o) H \mathbf{n} dS - \rho \iint_{S_B} (\boldsymbol{\alpha}^{(1)} \times \mathbf{n})[\Phi_t^{(1)} + g(\xi_3^{(1)} + \alpha_1^{(1)} y - \alpha_2^{(1)} x)] dS \\
& - \rho \iint_{S_B} [\frac{1}{2} \nabla\Phi^{(1)} \cdot \nabla\Phi^{(1)} + (\boldsymbol{\xi}^{(1)} + \boldsymbol{\alpha}^{(1)} \times \mathbf{x}) \cdot \nabla\Phi_t^{(1)}] \mathbf{n} dS - \rho g \iint_{S_B} (H\mathbf{x} \cdot k) \mathbf{n} dS \\
& + \frac{1}{2} \rho g \int_{WL} [\eta^{(1)} - (\xi_3^{(1)} + \alpha_1^{(1)} y - \alpha_2^{(1)} x)]^2 \sqrt{1 - n_z^2} dl \\
& - \rho g A_{wp} (\xi_3^{(2)} + \alpha_1^{(2)} y_f - \alpha_2^{(2)} x_f) k \\
& - \rho \iint_{S_B} \Phi_t^{(2)} \mathbf{n} dS
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
\mathbf{M}^{(2)} = & \ \tfrac{1}{2} \ \rho g \int_{wl} [\eta^{(1)} - (\xi_3^{(1)} + \alpha_1^{(1)} y - \alpha_2^{(1)} x)]^2 \sqrt{1 - n_z^2}\ (\mathbf{x} \times \mathbf{n}) dl \\
& - \rho \iint_{S_B} [\frac{1}{2} \nabla\Phi^{(1)} \cdot \nabla\Phi^{(1)} + (\boldsymbol{\xi}^{(1)} + \boldsymbol{\alpha}^{(1)} \times \mathbf{x}) \cdot \nabla\Phi_t^{(1)}] (\mathbf{x} \times \mathbf{n}) dS \\
& - \rho \iint_{S_B} (\boldsymbol{\xi}^{(1)} \times \mathbf{n})[\Phi_t^{(1)} + g(\xi_3^{(1)} + \alpha_1^{(1)} y - \alpha_2^{(1)} x)] dS \\
& - \rho g \iint_{S_B} \boldsymbol{\xi}^{(1)} \times (\boldsymbol{\alpha}^{(1)} \times \mathbf{n})(z + Z_o) dS \\
& - \rho \iint_{S_B} \boldsymbol{\alpha}^{(1)} \times (\mathbf{x} \times \mathbf{n})[\Phi_t^{(1)} + g(\xi_3^{(1)} + \alpha_1^{(1)} y - \alpha_2^{(1)} x)] dS \\
& - \rho g \iint_{S_B} (z + Z_o) H (\mathbf{x} \times \mathbf{n}) dS - \rho g \iint_{S_B} (H\mathbf{x} \cdot k)(\mathbf{x} \times \mathbf{n}) dS \\
& - \rho g i[-V\xi_2^{(2)} + A_{wp} y_f \xi_3^{(2)} + (Vz_b + L_{22})\alpha_1^{(2)} - L_{12}\alpha_2^{(2)} - Vx_b\alpha_3^{(2)}] \\
& - \rho g j[\ \ V\xi_1^{(2)} - A_{wp} x_f \xi_3^{(2)} - L_{12}\alpha_1^{(2)} + (Vz_b + L_{11})\alpha_2^{(2)} - Vy_b\alpha_3^{(2)}]
\end{aligned}
$$

$$- \quad \rho \iint_{S_B} (\mathbf{x} \times \mathbf{n}) \Phi_t^{(2)} dS$$

We apply (4.14) and their variations to the hydrostatic pressure integrals of (4.19) and make use of the following relations to simplify (4.19).

$$\begin{aligned}
-\rho \iint_{S_B} (\boldsymbol{\alpha}^{(1)} \times \mathbf{n})[\Phi_t^{(1)} &+ g(\xi_3^{(1)} + \alpha_1^{(1)} y - \alpha_2^{(1)} x)] dS \\
&= \boldsymbol{\alpha} \times F^{(1)} + \rho g \boldsymbol{\alpha} \times \iint_{S_B} (\boldsymbol{\alpha}^{(1)} \times \mathbf{n})(z + Z_o) dS \\
&= \boldsymbol{\alpha} \times F^{(1)} + \rho g V [-\alpha_1^{(1)} \alpha_3^{(1)} i - \alpha_2^{(1)} \alpha_3^{(1)} j + ((\alpha_1^{(1)})^2 + (\alpha_2^{(1)})^2) k]
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
-\rho \iint_{S_B} (\boldsymbol{\xi}^{(1)} \times \mathbf{n})[\Phi_t^{(1)} &+ g(\xi_3^{(1)} + \alpha_1^{(1)} y - \alpha_2^{(1)} x)] dS \\
&= \boldsymbol{\xi}^{(1)} \times F^{(1)} + \rho g \iint_{S_B} \boldsymbol{\xi}^{(1)} \times (\boldsymbol{\alpha}^{(1)} \times \mathbf{n})(z + Z_o) dS
\end{aligned}$$

$$\begin{aligned}
-\rho \iint_{S_B} \boldsymbol{\alpha}^{(1)} \times (\mathbf{x} \times \mathbf{n}) & [\Phi_t^{(1)} + g(\xi_3^{(1)} + \alpha_1^{(1)} y - \alpha_2^{(1)} x)] dS \\
= \boldsymbol{\alpha}^{(1)} \times M^{(1)} + \rho g \boldsymbol{\alpha}^{(1)} \times \{\iint_{S_B} & (\boldsymbol{\xi}^{(1)} \times \mathbf{n})(z + Z_o) dS + \iint_{S_B} [\boldsymbol{\alpha}^{(1)} \times (\mathbf{x} \times \mathbf{n})](z + Z_o) dS\} \\
= \boldsymbol{\alpha}^{(1)} \times M^{(1)} - \rho g V \boldsymbol{\alpha}^{(1)} \times (\boldsymbol{\xi}^{(1)} & \times k) - \rho g V \boldsymbol{\alpha}^{(1)} \times [\boldsymbol{\alpha}^{(1)} \times (y_b i - x_b j)]
\end{aligned}$$

The second-order force or moment due to $\Phi^{(2)}$ is decomposed into a part due to $\Phi_I^{(2)} + \Phi_S^{(2)}$ and the other part due to $\Phi_R^{(2)}$ as in the first-order. Then the force and moment (4.19) take forms

$$\begin{aligned}
\mathbf{F}^{(2)} &= \mathbf{F}_q + \mathbf{F}_p \\
&- \rho g A_{wp} (\xi_3^{(2)} + \alpha_1^{(2)} y_f - \alpha_2^{(2)} x_f) k \\
&- \rho \iint_{S_B} \mathbf{n} \frac{\partial \Phi_R^{(2)}}{\partial t} dS \\
\mathbf{M}^{(2)} &= \mathbf{M}_q + \mathbf{M}_p \\
&- \rho g i [-V \xi_2^{(2)} + A_{wp} y_f \xi_3^{(2)} + (V z_b + L_{22}) \alpha_1^{(2)} - L_{12} \alpha_2^{(2)} - V x_b \alpha_3^{(2)}] \\
&- \rho g j [V \xi_1^{(2)} - A_{wp} x_f \xi_3^{(2)} - L_{12} \alpha_1^{(2)} + (V z_b + L_{11}) \alpha_2^{(2)} - V y_b \alpha_3^{(2)}] \\
&- \rho \iint_{S_B} (\mathbf{x} \times \mathbf{n}) \frac{\partial \Phi_R^{(2)}}{\partial t} dS
\end{aligned} \tag{4.21}$$

where the subscript q denotes the force and moment due to the quadratic interaction of the first-order solution and p due to the second-order potential. The quadratic force and moment are defined by

$$
\begin{aligned}
\mathbf{F}_q &= \frac{1}{2}\rho g \int_{wl} [\eta^{(1)} - (\xi_3^{(1)} + \alpha_1^{(1)} y - \alpha_2^{(1)} x)]^2 \sqrt{1-n_z^2}\,\mathbf{n}dl \\
&- \rho \iint_{S_B} [\frac{1}{2}\nabla\Phi^{(1)}\cdot\nabla\Phi^{(1)} + (\boldsymbol{\xi}^{(1)} + \boldsymbol{\alpha}^{(1)}\times\mathbf{x})\cdot\nabla\Phi_t^{(1)}]\mathbf{n}dS \\
&+ \boldsymbol{\alpha}^{(1)} \times F^{(1)} - \rho g A_{wp}[\alpha_1^{(1)}\alpha_3^{(1)} x_f + \alpha_2^{(1)}\alpha_3^{(1)} y_f + \frac{1}{2}((\alpha_1^{(1)})^2 + (\alpha_2^{(1)})^2) Z_o] k
\end{aligned}
\tag{4.22}
$$

$$
\begin{aligned}
\mathbf{M}_q &= \frac{1}{2}\rho g \int_{wl} [\eta^{(1)} - (\xi_3^{(1)} + \alpha_1^{(1)} y - \alpha_2^{(1)} x)]^2 \sqrt{1-n_z^2}\;(\mathbf{x}\times\mathbf{n})dl \\
&- \rho \iint_{S_B} [\frac{1}{2}\nabla\Phi^{(1)}\cdot\nabla\Phi^{(1)} + (\boldsymbol{\xi}^{(1)} + \boldsymbol{\alpha}^{(1)}\times\mathbf{x})\cdot\nabla\Phi_t^{(1)}](\mathbf{x}\times\mathbf{n})dS \\
&+ \boldsymbol{\xi}^{(1)} \times F^{(1)} + \boldsymbol{\alpha}^{(1)} \times M^{(1)} \\
&+ \rho g(-V\xi_1^{(1)}\alpha_3^{(1)} + V\alpha_1^{(1)}\alpha_2^{(1)} x_b - V\alpha_2^{(1)}\alpha_3^{(1)} z_b - \frac{1}{2}V((a_1^{(1)})^2 - (a_3^{(1)})^2) y_b \\
&\qquad -\alpha_1^{(1)}\alpha_3^{(1)} L_{12} - \alpha_2^{(1)}\alpha_3^{(1)} L_{22} - \frac{1}{2}((\alpha_1^{(1)})^2 + (\alpha_2^{(1)})^2) Z_o A_{wp} y_f) i \\
&+ \rho g(-V\xi_2^{(1)}\alpha_3^{(1)} + V\alpha_1^{(1)}\alpha_3^{(1)} z_b + \frac{1}{2}V((a_2^{(1)})^2 - (a_3^{(1)})^2) x_b \\
&\qquad +\alpha_1^{(1)}\alpha_3^{(1)} L_{11} + \alpha_2^{(1)}\alpha_3^{(1)} L_{12} + \frac{1}{2}((\alpha_1^{(1)})^2 + (\alpha_2^{(1)})^2) Z_o A_{wp} x_f) j \\
&+ \rho g(V\xi_1^{(1)}\alpha_1^{(1)} + V\xi_2^{(1)}\alpha_2^{(1)} + V\alpha_2^{(1)}\alpha_3^{(1)} x_b - V\alpha_1^{(1)}\alpha_3^{(1)} y_b) k
\end{aligned}
$$

The second-order potential force and moment are given by

$$
\begin{aligned}
\mathbf{F}_p &= -\rho \iint_{S_B} \frac{\partial(\Phi_I^{(2)} + \Phi_S^{(2)})}{\partial t}\mathbf{n}dS \\
\mathbf{M}_p &= -\rho \iint_{S_B} (\mathbf{x}\times\mathbf{n}) \frac{\partial(\Phi_I^{(2)} + \Phi_S^{(2)})}{\partial t} dS
\end{aligned}
\tag{4.23}
$$

Adopting a form analogous to (1.12) the second-order force $\mathbf{F}^{(2)}$ can be expressed as (the moment takes an identical form),

$$
\mathbf{F}^{(2)} = \mathrm{Re}\sum_i \sum_j \mathbf{F}_{ij}^{+} \mathrm{e}^{\mathrm{i}(\omega_i+\omega_j)t} + \mathbf{F}_{ij}^{-} \mathrm{e}^{\mathrm{i}(\omega_i-\omega_j)t} \tag{4.24}
$$

4.1.7 Haskind Exciting Force and Indirect Second-order Force

The linear wave exciting force and the second-order potential force F_p may be evaluated not from the scattering solution but from the appropriate component of the radiation solution. The first-order force evaluated in this way is referred to as the Haskind exciting force and the second-order force as the indirect force.

4.1.8 Haskind exciting force

We consider a complex amplitude of a spectral component of the first-order wave exciting force. The force for the mode j is given by

$$F_j = -i\omega\rho \iint_{S_B} n_j(\phi_I + \phi_S) dS \tag{4.25}$$

From Green's theorem, applied to ϕ_S and a component of the radiation potential ϕ_j, we have

$$0 = \iint_{S_B} \left(\frac{\partial \phi_j}{\partial n}\phi_S - \frac{\partial \phi_S}{\partial n}\phi_j\right) dS \tag{4.26}$$

It can be shown that integrals over the free surface, on the bottom and at the far field vanish due to the boundary conditions for ϕ_S and ϕ_j on these surfaces. (Newman (1977, Section 6.18)) Thus they do not appear in (4.26).

Upon substituting (4.26) into (4.25) with the conditions (2.10) and (2.11), the Haskind relation follows in the form

$$F_j = -i\omega\rho \iint_{S_B} \left(n_j\phi_I - \frac{\partial \phi_I}{\partial n}\phi_j\right) dS \tag{4.27}$$

4.1.9 Indirect second-order force

We consider a complex amplitude of a spectral component of the second-order potential force F_p. The force for the mode j is given by

$$F_{pj}^{\pm} = -i(\omega_i \pm \omega_j)\rho \iint_{S_B} n_j(\phi_I^{\pm} + \phi_S^{\pm}) dS \tag{4.28}$$

From Green's theorem, applied to $\phi_S^{\pm}$ and $\phi_j^{\pm}$, we have

$$0 = \iint_{S_B} \left(\frac{\partial \phi_j^{\pm}}{\partial n}\phi_S^{\pm} - \frac{\partial \phi_S^{\pm}}{\partial n}\phi_j^{\pm}\right) dS + \iint_{S_F} \left(\frac{\partial \phi_j^{\pm}}{\partial z}\phi_S^{\pm} - \frac{\partial \phi_S^{\pm}}{\partial z}\phi_j^{\pm}\right) dS \tag{4.29}$$

It can be shown that the integrals on the sea bottom and at the far field vanish due to the boundary conditions. Upon substituting (3.18-21) into (4.29), we have

$$\iint_{S_B} n_j\phi_S^{\pm} dS = \iint_{S_B} Q_B^{\pm}\phi_j^{\pm} dS + \frac{1}{g}\iint_{S_F} (Q_{IB}^{\pm} + Q_{BB}^{\pm})\phi_j^{\pm} dS \tag{4.30}$$

Thus the force can be obtained from

$$F_{pj}^{\pm} = -i(\omega_i \pm \omega_j)\rho\left[\iint_{S_B} (n_j\phi_I^{\pm} + Q_B^{\pm}\phi_j^{\pm}) dS + \frac{1}{g}\iint_{S_F} (Q_{IB}^{\pm} + Q_{BB}^{\pm})\phi_j^{\pm} dS\right] \tag{4.31}$$

The radiation potential $\phi_j^{\pm}$ with the frequency $\omega_i \pm \omega_j$ is often referred to as an assisting potential.

4.2 Equations of Motion

The translational and rotational motions of the body are governed by

$$m\hat{\mathbf{x}}_{gtt} = F_T \tag{4.32}$$

and

$$L_t = M_T \tag{4.33}$$

In (4.32), $\hat{\mathbf{x}}_{gtt}$ is the acceleration of the center of mass of the body and F_T is the total force due to the fluid pressure, the body mass and the external force. In (4.33), M_T is the total moment about the $\hat{\mathbf{x}}_g$ and L_t is the time rate of change of the angular momentum in the $\hat{\mathbf{x}}$ coordinate system.

It is convenient to express (4.33) in the body-fixed coordinate system, $\mathbf{x}$, in which the moment of inertia is time invariant. Following Ogilvie(1983),

$$\mathbf{I}_g\boldsymbol{\omega}_t + \boldsymbol{\omega} \times \mathbf{I}_g\boldsymbol{\omega} = TM_T \tag{4.34}$$

where $\mathbf{I}_g$ is the moment of inertia about the center of mass, $\boldsymbol{\omega}_t(\boldsymbol{\omega})$ the angular acceleration (velocity) and T the tranform matrix (4.3).

If we are interested in the motion about the origin of the body coordinate system, $\mathbf{x}_o$, which may differ from the center of mass, the following relations are to be substitued into (4.34).

$$\mathbf{I}_g\boldsymbol{\omega} = \mathbf{I}\boldsymbol{\omega} - m\mathbf{x}_g \times (\boldsymbol{\omega} \times \mathbf{x}_g) \tag{4.35}$$

and

$$M_T = M_o - \hat{\mathbf{x}}_g \times F_T \tag{4.36}$$

where $\mathbf{I}$ and M_o are the moment of inertia and moment about o.

(4.32) and (4.34) with (4.35-6), describe the motion of the rigid body.

To derive the first- and the second-order equations of motion, we express $\hat{\mathbf{x}}_g$, F_T, TM_T and $\boldsymbol{\omega}$ in perturbation series.

$$\hat{\mathbf{x}}_g = \mathbf{x}_g + \boldsymbol{\xi} + \boldsymbol{\alpha} \times \mathbf{x}_g + H\mathbf{x}_g + O(A^3) \tag{4.37}$$

$$F = F^{(1)} + F_B^{(1)} + F_E^{(1)} + F^{(2)} + F_B^{(2)} + F_E^{(2)} + O(A^3) \tag{4.38}$$

$$\begin{aligned} TM_T \;=\;& M^{(1)} - \mathbf{x}_g \times F^{(1)} + M^{(2)} - \mathbf{x}_g \times F^{(2)} - \boldsymbol{\xi}^{(1)} \times F^{(1)} - \boldsymbol{\alpha}^{(1)} \times M^{(1)} \\ +\;& \mathbf{x}_g \times (\boldsymbol{\alpha}^{(1)} \times F^{(1)}) + \Gamma^{(1)} + \Gamma^{(2)} + O(A^3) \end{aligned} \tag{4.39}$$

In (4.38-39), the subscript $_B$ denotes the gravitational force on the body mass, $_E$ external force which is assumed to be expandable in a perturbation series. The force without subscript is the hydro-static and -dynamic force derived in Sections 4.1 and 4.2. $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are the first- and second-order components of

$$\Gamma = T[M_B^{(1)} + M_B^{(2)} + M_E^{(1)} + M_E^{(2)} - \hat{\mathbf{x}}_g \times (F_E^{(1)} + F_E^{(2)}) + O(A^3)] \tag{4.40}$$

Equilibrium, when the body is at rest, leads to the conditions

$$F^{(0)} + F_B^{(0)} + F_E^{(0)} = 0 \text{ and } M^{(0)} + M_B^{(0)} + M_E^{(0)} = 0 \tag{4.41}$$

The moment M_E consists of a pure torque M_τ and the moment due to the external force

$$M_E = M_\tau + \hat{\mathbf{x}}_e \times F_E \tag{4.42}$$

The angular velocity in the $\hat{\mathbf{x}}$ coordinate system can be obtained by linear superposition of the contributions from time derivatives of Euler-angles.

$$\hat{\boldsymbol{\omega}} = \begin{pmatrix} \alpha_{1t} \\ 0 \\ 0 \end{pmatrix} + T_1^t \begin{pmatrix} 0 \\ \alpha_{2t} \\ 0 \end{pmatrix} + T_1^t T_2^t \begin{pmatrix} 0 \\ 0 \\ \alpha_{3t} \end{pmatrix} \tag{4.43}$$

and angular velocity in the $\mathbf{x}$ coordinate system, $\boldsymbol{\omega}$, is obtained from

$$\boldsymbol{\omega} = T^t \hat{\boldsymbol{\omega}} \tag{4.44}$$

Thus we have at the first-order

$$\boldsymbol{\omega}^{(1)} = \begin{pmatrix} \alpha_{1t}^{(1)} \\ \alpha_{2t}^{(1)} \\ \alpha_{3t}^{(1)} \end{pmatrix} \tag{4.45}$$

and at the second-order

$$\boldsymbol{\omega}^{(2)} = \begin{pmatrix} \alpha_{1t}^{(2)} + \alpha_{2t}^{(1)} \alpha_3^{(1)} \\ \alpha_{2t}^{(2)} - \alpha_{1t}^{(1)} \alpha_3^{(1)} \\ \alpha_{3t}^{(2)} + \alpha_{1t}^{(1)} \alpha_2^{(1)} \end{pmatrix} \tag{4.46}$$

Upon substitution of (4.35-46) into (4.32) and (4.34), the equations of motion at each order follows. These are given by

$$[(M + M^E + A)\left\{ \begin{matrix} \boldsymbol{\xi}_{tt}^{(1)} \\ \boldsymbol{\alpha}_{tt}^{(1)} \end{matrix} \right\} + (B + B^E)\left\{ \begin{matrix} \boldsymbol{\xi}_{t}^{(1)} \\ \boldsymbol{\alpha}_{t}^{(1)} \end{matrix} \right\} + (C + C^E)\left\{ \begin{matrix} \boldsymbol{\xi}^{(1)} \\ \boldsymbol{\alpha}^{(1)} \end{matrix} \right\}] = \left\{ \begin{matrix} F_{exc}^{(1)} \\ M_{exc}^{(1)} \end{matrix} \right\} \tag{4.47}$$

and

$$[(M + M^E + A)\left\{ \begin{matrix} \boldsymbol{\xi}_{tt}^{(2)} \\ \boldsymbol{\alpha}_{tt}^{(2)} \end{matrix} \right\} + (B + B^E)\left\{ \begin{matrix} \boldsymbol{\xi}_{t}^{(2)} \\ \boldsymbol{\alpha}_{t}^{(2)} \end{matrix} \right\} + (C + C^E)\left\{ \begin{matrix} \boldsymbol{\xi}^{(2)} \\ \boldsymbol{\alpha}^{(2)} \end{matrix} \right\}] = \left\{ \begin{matrix} F_{exc}^{(2)} \\ M_{exc}^{(2)} \end{matrix} \right\} \tag{4.48}$$

where A and B are the added mass and damping coefficients and they depend on the frequency. M^E, B^E and C^E are the external force components which are proportional to the unknown body acceleration, velocity and the motion amplitude, respectively. They may or may not be different for the first and the second-order motions. M and C are the same in (4.47) and (4.48) and they are given by

$$M = \begin{bmatrix} m & 0 & 0 & 0 & mz_g & -my_g \\ 0 & m & 0 & -mz_g & 0 & mx_g \\ 0 & 0 & m & my_g & -mx_g & 0 \\ 0 & -mz_g & my_g & I_{11} & I_{12} & I_{13} \\ mz_g & 0 & -mx_g & I_{21} & I_{22} & I_{23} \\ -my_g & mx_g & 0 & I_{31} & I_{32} & I_{33} \end{bmatrix} \tag{4.49}$$

$$C = g \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho A_{wp} & \rho A_{wp} y_f & -\rho A_{wp} x_f & 0 \\ 0 & 0 & \rho A_{wp} y_f & \rho(V z_b + L_{22}) - mz_g & -\rho L_{12} & -\rho V x_b + mx_g \\ 0 & 0 & -\rho A_{wp} x_f & -\rho L_{12} & \rho(V z_b + L_{11}) - mz_g & -\rho V y_b + my_g \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.50}$$

Finally the force and moment on the right-hand side of (4.47) are

$$\begin{aligned} F_{exc}^{(1)} &= -\rho \iint_{S_B} \Phi_t^{D(1)} \mathbf{n} dS + (\Sigma F_E^{(1)})_{ex} \\ M_{exc}^{(1)} &= -\rho \iint_{S_B} (\mathbf{x} \times \mathbf{n}) \Phi_t^{D(1)} dS + (\Sigma M_\tau^{(1)} + \Sigma \mathbf{x}_e \times F_E^{(1)})_{ex} \end{aligned} \tag{4.51}$$

and those on the right-hand side of (4.48) are

$$\begin{aligned} F_{exc}^{(2)} = \quad & F_q + F_p - mH_{tt}\mathbf{x}_g + (\Sigma F_E^{(2)})_{ex} \\ M_{exc}^{(2)} = \quad & M_q + M_p - \boldsymbol{\xi}^{(1)} \times F^{(1)} - \boldsymbol{\alpha}^{(1)} \times M^{(1)} \\ + \quad & \rho g[V\xi_1^{(1)}\alpha_3^{(1)} i + V\xi_2^{(1)}\alpha_3^{(1)} j - (V\xi_1^{(1)}\alpha_1^{(1)} - V\xi_2^{(1)}\alpha_2^{(1)})k] \\ + \quad & mg[(\frac{1}{2}((\alpha_1^{(1)})^2 - (\alpha_3^{(1)})^2)y_g - \alpha_1^{(1)}\alpha_2^{(1)}x_g + \alpha_2\alpha_3 z_g)i \\ - \quad & (\frac{1}{2}((\alpha_2^{(1)})^2 - (\alpha_3^{(1)})^2)x_g + \alpha_1\alpha_3 z_g)j + (\alpha_1\alpha_3 y_g - \alpha_2\alpha_3 x_g)k] \\ + \quad & \mathbf{x}_g \times (\boldsymbol{\alpha}^{(1)} \times F^{(1)}) - m\mathbf{x}_g \times H_{tt}\mathbf{x}_g - \mathbf{I}_g \boldsymbol{\alpha}_{tt}^q \\ - \quad & \boldsymbol{\alpha}_t^{(1)} \times \mathbf{I}_g \boldsymbol{\alpha}_t^{(1)} + \Sigma[H\mathbf{x}_e \times F_E^{(0)} - \boldsymbol{\alpha}^{(1)} \times ((\boldsymbol{\alpha}^{(1)} \times \mathbf{x}_e) \times F_E^{(0)})] \\ + \quad & \{\Sigma[M_\tau^{(2)} + \mathbf{x}_e \times F_E^{(2)} - \boldsymbol{\alpha}^{(1)} \times M_\tau^{(1)} - (\mathbf{x}_e - \mathbf{x}_g) \times (\boldsymbol{\alpha}^{(1)} \times F_E^{(1)})]\}_{ex} \end{aligned} \tag{4.52}$$

The subscript $_{ex}$ denotes the part of the force and moment which is not linearly proportional to the motion amplitude, the velocity or the acceleration at each order. In (4.52), $\boldsymbol{\alpha}_{tt}^{q}$ is a vector and its components are the time derivatives of the second terms of the vector elements of (4.46).

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APPENDICES

NOTES ON BODIES OR PANELS IN THE FREE SURFACE

JNN - September 1993

1. INTRODUCTION

These notes document recent work intended to (1) extend WAMIT to bodies where part or all of the submerged surface is in the plane of the free surface, with zero draft; and (2) removal of irregular frequencies by adding panels in the free surface interior to the body with an imposed homogeneous Neumann (or Dirichlet) condition. The latter development was intended to follow the theory in the report by Kleinman, denoted here by [K], but differences and questions have arisen which are the principal motivation for the notes.

The notation of the WAMIT User Manual is followed. The most important differences from [K] are the definition of the unit normal vector, here taken to be into the body and out of the fluid domain, and the Green function G, defined here by equation (2.4)

$$G(\mathbf{x};\boldsymbol{\xi}) = \frac{1}{r} + \frac{1}{r'} + \frac{2K}{\pi}\int_0^\infty dk\, \frac{e^{k(z+\zeta)}}{k-K} J_0(kR) \tag{1.1}$$

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 \tag{1.2}$$

$$r'^2 = (x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2, \tag{1.3}$$

where $J_0(x)$ is the Bessel function of zero order. The correponding function γ in [K] is related to G via the equation $G = -2\pi\gamma$. For simplicity the depth is assumed infinite.

The velocity potential ϕ is governed by Laplace's equation in the fluid domain, subject to a prescribed Neumann condition $\phi_n = V$ on the body surface S_b, the linear free surface condition

$$\phi_z - K\phi = 0, \qquad \text{on} \;\; z = 0 \tag{1.4}$$

and a radiation condition at infinity. The wavenumber is $K = \omega^2/g$, and g is the acceleration of gravity.

The Green function G satisfies the free surface condition (1.4) with respect to both coordinate systems. More explicitly,

$$G_z - KG = 0, \qquad \text{on} \;\; z = 0 \tag{1.5}$$

$$G_\zeta - KG = 0, \qquad \text{on} \;\; \zeta = 0 \tag{1.6}$$

It is not obvious that these relations can be applied in the limit when $r = r' = 0$, i.e. when the source and field points coincide on the free surface. In the vicinity of this singular point it is known that G is of the form

$$G(\mathbf{x};\boldsymbol{\xi}) = \frac{1}{r} + \frac{1}{r'} - 2Ke^{K(z+\zeta)}\left(\log(r' + |z+\zeta|) + (\gamma - \log 2) + r' + \mathrm{O}(r'^2 \log r')\right) \tag{1.7}$$

If this expansion is substituted in (1.5) or (1.6), it can be confirmed that the same relations apply even at the singular point. Thus there is no 'delta function' in the free-surface conditions (1.5) and (1.6).

2 CONVENTIONAL INTEGRAL EQUATION FOR THE VELOCITY POTENTIAL

In the 'normal' case of a floating body, the submerged surface S_b is entirely below the plane $z = 0$, except for a line of intersection of the two surfaces. The fluid domain D_+ is exterior to the body, below the free surface, and extends to infinity horizontally and vertically. The complementary domain D_- is inside the body, below the plane $z = 0$.

With the above definitions, it follows from Green's theorem that

$$2\pi\alpha(\mathbf{x})\phi(\mathbf{x}) + \iint_{S_b} \phi(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} = \iint_{S_b} V(\boldsymbol{\xi}) G d\boldsymbol{\xi}. \tag{2.1}$$

Here, as in [K], α is equal to 2 for points in D_+, 1 for points on S_b, and 0 for points in D_-.

The definitions of α in (2.1) are consistent with the 'jump conditions' corresponding to the singularity $1/r$ in G. Thus, in the limits where the point $\mathbf{x}$ approaches S_b from $D_\pm$,

$$I_\pm(\phi) = \lim_{\mathbf{x}\to S_\pm} \iint_{S_b} \phi(\boldsymbol{\xi}) \frac{\partial}{\partial n_\xi} \frac{1}{r} d\boldsymbol{\xi} = \mp 2\pi\phi + ⨍⨍_{S_b} \phi(\boldsymbol{\xi}) \frac{\partial}{\partial n_\xi} \frac{1}{r} d\boldsymbol{\xi} \tag{2.2}$$

Here the symbol $⨍⨍$ denotes, as in the simpler Cauchy principal-value integral, that a surface of vanishingly small area including the singular point $r = 0$ is excluded from the integral. This is the appropriate interpretation of the integral on the left side of (2.1) for the case where $\mathbf{x}$ is on S_b. Thus (2.1) is rewritten in the more explicit form

$$2\pi\phi(\mathbf{x}) + ⨍⨍_{S_b} \phi(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} = \iint_{S_b} V(\boldsymbol{\xi}) G d\boldsymbol{\xi} \qquad (\mathbf{x} \in S_b) \tag{2.3}$$

3 INTEGRAL EQUATIONS FOR S_b IN THE FREE SURFACE

The objective here is to extend (2.1) to the case where part or all of the body surface coincides with the free surface. Examples of such bodies are a floating caisson where the interior 'roof' is in the plane $z = 0$, and a 'cicular dock' consisting of a disk of zero draft.

In these cases, for those portions of S_b lying in the free surface, the two Rankine singularities in the Green function coalesce, and the effect is to double the jump in (2.2). However caution is required to correctly account for the jump when the image source $1/r'$ is included. For completeness, in relation to the corresponding equations [K, 51b-52b], we write the following derivatives and limits:

$$\frac{\partial}{\partial z}\Big(\frac{1}{r}\Big) = \frac{-(z-\zeta)}{r^3} \tag{3.1a}$$

$$\frac{\partial}{\partial z}\Big(\frac{1}{r'}\Big) = \frac{-(z+\zeta)}{r^3} \tag{3.1b}$$

$$\frac{\partial}{\partial \zeta}(\frac{1}{r}) = \frac{(z-\zeta)}{r^3} \tag{3.1c}$$

$$\frac{\partial}{\partial \zeta}(\frac{1}{r'}) = \frac{-(z+\zeta)}{r^3} \tag{3.1d}$$

Thus

$$\frac{\partial}{\partial z}(\frac{1}{r} + \frac{1}{r'}) = \frac{-2z}{r^3} \qquad \text{on} \quad \zeta = 0 \tag{3.1e}$$

$$\frac{\partial}{\partial \zeta}(\frac{1}{r} + \frac{1}{r'}) = 0 \qquad \text{on} \quad \zeta = 0 \tag{3.1f}$$

In the limit $z \to 0-$ (3.1e) tends to a delta function with area 4π, but (3.1f) is identically zero and does not contribute a jump. The former is consistent with [K] equation (51b), but the latter contradicts the jump in (52b).

I will proceed assuming (3.1f) is correct. Two alternative derivations are given for the case where S_b coincides with the plane $z = 0$.

First start with (2.1), with the field point in the fluid domain, below the body surface, and approach the surface at a point where the surface coincides with the plane $z = 0$. Prior to the limit, the free term in (2.1) has the value $4\pi\phi$. In the limit the same value applies, since the singular contribution to the kernel from the two Rankine singularties is

$$\frac{\partial}{\partial n_\xi}(\frac{1}{r} + \frac{1}{r'})_{S_b} = \frac{\partial}{\partial \zeta}(\frac{1}{r} + \frac{1}{r'})_{\zeta=0} = 0 \tag{3.1}$$

Thus, (2.1) is applicable to such a body surface with the understanding that $\alpha = 2$ when the point $\mathbf{x}$ is on the body surface, in the plane $z = 0$. Note in this case that the unit normal is positive upwards, pointing from the fluid domain toward the complementary domain above the free surface. In the case of a body which is entirely in the plane $z = 0$, the resulting integral equation is

$$4\pi\phi(\mathbf{x}) + \int\!\!\!\!\!\!-\!\!\!\!\int_{S_b} \phi(\boldsymbol{\xi}) \frac{\partial G}{\partial \zeta} d\boldsymbol{\xi} = \iint_{S_b} V(\boldsymbol{\xi}) G d\boldsymbol{\xi}. \qquad (\mathbf{x} \in S_b) \tag{3.2}$$

As an alternative approach, consider a body with small finite draft T, e.g. with a flat bottom and vertical sides. In this case (2.3) is applicable on the exact body. Neglecting the contribution from the vertical sides, the contribution from the image source to the left-hand side of (2.3) is

$$\iint_{S_b} \phi(\boldsymbol{\xi}) \frac{\partial}{\partial \zeta}(\frac{1}{r'}) d\boldsymbol{\xi} = \iint_{S_b} \phi(\boldsymbol{\xi}) \frac{-(z+\zeta)}{r^3} d\boldsymbol{\xi} \qquad \text{on} \quad z = \zeta = -T \tag{3.2a}$$

In the limit $T \to 0$ the kernel tends to a delta function with area 2π, augmenting the free term in (2.3) and thus in agreement with (3.2).

Using the free-surface boundary conditions (1.5 - 1.6), two equivalent integral equations may be written in the forms

$$4\pi\phi(\mathbf{x}) + \int\!\!\!\!\!\!-\!\!\!\!\int_{S_b} \phi(\boldsymbol{\xi}) \frac{\partial G}{\partial z} d\boldsymbol{\xi} = \iint_{S_b} V(\boldsymbol{\xi}) G d\boldsymbol{\xi}. \qquad (\mathbf{x} \in S_b) \tag{3.3}$$

and

$$4\pi\phi(\mathbf{x}) + K\iint_{S_b} \phi(\boldsymbol{\xi}) G d\boldsymbol{\xi} = \iint_{S_b} V(\boldsymbol{\xi}) G d\boldsymbol{\xi}. \qquad (\mathbf{x} \in S_b) \tag{3.4}$$

Equation (3.4) is most effective for numerical computations since the logarithmic singularity in (1.7) is not amplified by differentiation. WAMIT has been modified in accordance with (3.4), and used to solve the circular dock problem. The results appear to be consistent. It is essential in this case to use the 'ILOG=1' option to integrate the logarithmic singularity in (1.7) analytically over each panel in the free surface.

Presumably the integral equations (3.2-4) have no homogeneous solutions, or irregular frequencies. This can be inferred from the fact that the interior volume between S_b and the free surface is zero.

4 EXISTENCE OF IRREGULAR FREQUENCIES

Hereafter the body is assumed to have nonzero submerged volume, to intersect the free surface normally, and no portion of its surface lies in the plane $z = 0$ except for the normal intersection along a waterline contour. Define S_i as the portion of the plane $z = 0$ interior to S_b. The union of S_b and S_i is the closed surface S, with the unit normal defined in a consistent manner to point into the interior volume D_-. Define an interior potential ϕ' which is harmonic in D_-, and satisfies the free surface condition (1.4) on S_i. Applying Green's theorem in the usual manner,

$$2\pi\big(\alpha(\mathbf{x}) - 2\big)\phi'(\mathbf{x}) + \iint_{S_b} \phi'(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} = \iint_{S_b} V'(\boldsymbol{\xi}) G d\boldsymbol{\xi}. \tag{4.1}$$

Here $V' = \phi'_n$ is the corresponding normal velocity on S_b.

Now assume that the interior potential ϕ' satisfies the homogeneous Dirichlet condition $\phi' = 0$ on S_b. It is known that nontrivial solutions exist for certain body shapes at discrete eigenfrequencies, and it is assumed that the same is true for general bodies. Physically, these are analogous to 'sloshing modes' corresponding to standing waves inside the body, except that the homogeneous body boundary condition is Dirichlet instead of Neumann. For such solutions the normal velocity V' is generally nonzero on S_b, but from (4.1) and the boundary condition $\phi' = 0$ on S_b it follows that

$$\iint_{S_b} V'(\boldsymbol{\xi}) G d\boldsymbol{\xi} = 0 \qquad (\mathbf{x} \in S_b \cup D_+) \tag{4.2}$$

This proves that there is a discrete set of eigenfrequencies (i.e. the irregular frequencies) where the corresponding normal velocity V' is orthogonal to G on the body.

Next we define an external 'radiation' potential ϕ which satisfies the usual conditions outside the body, as in §1, with the specified normal derivative $\phi_n = V = V'$ on S_b. This is essentially a radiation problem with a special distribution of normal velocity on the body. The solution is assumed to exist and to be nontrivial and unique, just as for any more conventional radiation (or scattering) problem. However for this particular external potential the right-hand side of (2.3) will vanish when $\mathbf{x}$ is on S_b. It follows that ϕ is a homogeneous solution of (2.3). This proves the existence of irregular frequencies in the context of the exterior potential problem.

5 EXTENDED-BOUNDARY INTEGRAL EQUATIONS

In [K] it is shown that while (2.3) may have homogeneous solutions at the irregular frequencies, there is (at most) one solution of the integral equation (2.1) in the extended domain $S = S_b \cup S_i$. Thus we augment (2.3) by the additional equation

$$\iint_{S_b} \phi(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} = \iint_{S_b} V(\boldsymbol{\xi}) G d\boldsymbol{\xi} \qquad (\mathbf{x} \in S_i) \tag{5.1}$$

A more useful second-kind equation [K, 86] is derived which, in the present notation, takes the form

$$2\pi\Phi(\mathbf{x}) + \iint_{S_b} \Phi(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} + \frac{1}{2} \iint_{S_i} \Phi(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} = \iint_{S_b} V(\boldsymbol{\xi}) G d\boldsymbol{\xi} \qquad (\mathbf{x} \in S_b \cup S_i) \tag{5.2}$$

In [K] it is shown (using the questionable jump relation (52b)) that (5.2) has (at most) only one solution, and that this solution is equal to ϕ on S_b.

At a more pragmatic level, our numerical experience indicates that the appropriate replacement for (5.2) is

$$2\pi\Phi(\mathbf{x}) + \iint_{S_b} \Phi(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} - \frac{1}{2} \iint_{S_i} \Phi(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} = \iint_{S_b} V(\boldsymbol{\xi}) G d\boldsymbol{\xi} \qquad (\mathbf{x} \in S_b \cup S_i) \tag{5.3}$$

The only difference is the sign preceding the integral over S_i.

In fact, the numerical tests are performed with a modified version of WAMIT where the influence functions correspond to the slightly different integral equations

$$2\pi\Phi(\mathbf{x}) + \iint_{S_b} \Phi(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} + \iint_{S_i} \Phi(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} = \iint_{S_b} V(\boldsymbol{\xi}) G d\boldsymbol{\xi} \qquad (\mathbf{x} \in S_b) \tag{5.4a}$$

$$-4\pi\Phi(\mathbf{x}) + \iint_{S_b} \Phi(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} - K \iint_{S_i} \Phi(\boldsymbol{\xi}) G d\boldsymbol{\xi} = \iint_{S_b} V(\boldsymbol{\xi}) G d\boldsymbol{\xi} \qquad (\mathbf{x} \in S_i) \tag{5.4b}$$

Equations (5.3) and (5.4) are equivalent after multiplying the unknown Φ by a factor -2 in the domain of S_i, and using (1.6) to replace the normal derivative of G on S_i in (5.4b). This should be borne in mind in judging the relevance of the statement that the numerical tests support (5.3a-b), but not (5.2).

Note that the first and third terms in (5.4b) are precisely the same as (minus) the left side of (3.2), suggesting a possible connection with the absence of irregular frequencies in the dock problem.

Finally we want to show that there are no homogeneous solutions of (5.4), that is no nontrivial solutions Φ_0 of the pair of integral equations

$$2\pi\Phi_0(\mathbf{x}) + \iint_{S_b} \Phi_0(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} + \iint_{S_i} \Phi_0(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} = 0 \qquad (\mathbf{x} \in S_b) \tag{5.5a}$$

$$-4\pi\Phi_0(\mathbf{x}) + \iint_{S_b} \Phi_0(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} + \iint_{S_i} \Phi_0(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} = 0 \qquad (\mathbf{x} \in S_i) \tag{5.5b}$$

For this purpose, asume that Φ_0 is a nontrivial solution of (5.5) and define the following two potentials (which together are used in place of [K 88]):

$$\phi_b = \iint_{S_b} \Phi_0(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} \qquad (\mathbf{x} \in D_-) \tag{5.6a}$$

$$\phi_i = \iint_{S_i} \Phi_0(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} = -K \iint_{S_i} \Phi_0 G d\boldsymbol{\xi} \qquad (\mathbf{x} \in D_- \cup S_b \cup D_+) \tag{5.6b}$$

The following conditions apply on S_b:

$$\phi_b = 2\pi\Phi_0(\mathbf{x}) + \fint\!\!\!\fint_{S_b} \Phi_0(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} \qquad (\mathbf{x} \in S_b) \tag{5.7a}$$

$$\phi_i = \iint_{S_i} \Phi_0(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} \qquad (\mathbf{x} \in S_b) \tag{5.7b}$$

Thus, from (5.5a),

$$\phi_b + \phi_i = 0 \qquad (\mathbf{x} \in S_b) \tag{5.8}$$

The following conditions apply as $z \to 0-$ on S_i:

$$\phi_b = \iint_{S_b} \Phi_0(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} \qquad (\mathbf{x} \in S_i) \tag{5.9a}$$

$$\phi_i = \fint\!\!\!\fint_{S_i} \Phi_0(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} \qquad (\mathbf{x} \in S_i) \tag{5.9b}$$

The free-surface condition (1.5) can be invoked to show that

$$\phi_{bz} = K\phi_b = K \iint_{S_b} \Phi_0(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} \qquad (\mathbf{x} \in S_i) \tag{5.10a}$$

where (5.9a) is used in the latter relation. For the analogous derivative of ϕ_i, the jump associated with (3.1e) gives an extra contribution and it follows from the last form of (5.6b) that

$$\phi_{iz} = K\phi_i - 4\pi K\Phi_0 = K \fint\!\!\!\fint_{S_i} \Phi_0(\boldsymbol{\xi}) \frac{\partial G}{\partial n_\xi} d\boldsymbol{\xi} - 4\pi K\Phi_0 \qquad (\mathbf{x} \in S_i) \tag{5.10b}$$

Combining (5.10a) and (5.10b), and invoking (5.5b),

$$\phi_{bz} + \phi_{iz} = 0 \qquad (\mathbf{x} \in S_i) \tag{5.11}$$

¿From (5.8) and (5.11), and the uniqueness proof for solutions of Laplace's equation with combined Dirichlet and Neumann boundary conditions, it follows that

$$\phi_b + \phi_i = 0 \qquad (\mathbf{x} \in D_-) \tag{5.12}$$

The remainder of the proof follows as in [K 89-95].

WAMIT V3.1S (Second-Order Module) – Theoretical Background

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1. INTRODUCTION

The WAMIT Second-Order Module, currently Version 3.1S, is designed to solve the second-order boundary-value problem for the interaction of monchromatic or bichromatic incident waves with a prescribed body. As in the first-order program (Version 3.1ID), the body is defined by an ensemble of panels. In the second-order solution the boundary conditions on both the body and free surface must be re-considered, with particular care required in the case of the free surface. This document is intended to describe the corresponding analysis in sufficient detail to explain the logic of the program. As the second-order module is extended and refined this document will be modified in parallel. The date and version number above should be referenced in this context.

In the notation to be followed the subscripts i, j, k are used to denote the frequencies of different linear solutions, and ℓ, m, n are used to denote the Fourier components of the same solutions, respectively. κ will be used for the wavenumber, to distinguish from the integer subscript k. First-order components such as the velocity potentials ϕ_j in (1.2) are distinguished from second-order components ϕ_{ij} by the number of subscripts.

The velocity potential is expanded in the form

$$\Phi(\mathbf{x}, t) = \epsilon\Phi^{(1)}(\mathbf{x}, t) + \epsilon^2\Phi^{(2)}(\mathbf{x}, t) + \cdots \tag{1.1}$$

where $\mathbf{x}$ is a fixed Cartesian coordinate system, and t denotes time. Assuming a discrete spectrum with frequency components $\omega_j > 0$,

$$\Phi^{(1)}(\mathbf{x}, t) = \mathrm{Re}\sum_j \phi_j(\mathbf{x})e^{i\omega_j t} \tag{1.2}$$

$$\Phi^{(2)}(\mathbf{x}, t) = \mathrm{Re}\sum_i \sum_j \left[\phi_{ij}^{+}(\mathbf{x})e^{i(\omega_i+\omega_j)t} + \phi_{ij}^{-}(\mathbf{x})e^{i(\omega_i-\omega_j)t}\right] \tag{1.3}$$

The second-order potentials $\phi_{ij}^{\pm}$ can be defined to satisfy the symmetry relations

$$\phi_{ij}^{+} = \phi_{ji}^{+} \quad \text{and} \quad \phi_{ij}^{-} = \phi_{ji}^{-*} \tag{1.4}$$

since anti-symmetric components will not contribute to (1.3).

The free-surface boundary conditions satisfied by these potentials are

$$\frac{\partial^2 \Phi^{(1)}}{\partial t^2} + g \frac{\partial \Phi^{(1)}}{\partial z} = 0 \tag{1.5}$$

$$\frac{\partial^2 \Phi^{(2)}}{\partial t^2} + g \frac{\partial \Phi^{(2)}}{\partial z} = Q(x, y; t) \tag{1.6}$$

on $z = 0$. Here the inhomogeneous right-hand-side of the second-order free-surface condition (1.6) defines the quadratic forcing function

$$Q = \frac{1}{g} \frac{\partial \Phi^{(1)}}{\partial t} \frac{\partial}{\partial z} \left(\frac{\partial^2 \Phi^{(1)}}{\partial t^2} + g \frac{\partial \Phi^{(1)}}{\partial z} \right) - \frac{\partial}{\partial t} (\nabla \Phi^{(1)} \cdot \nabla \Phi^{(1)}) \tag{1.7}$$

where the right-hand-side is to be evaluated on $z = 0$. In the following evaluations of second-order products of first-order oscillatory quantities use is made of the relation

$$\mathrm{Re}(A \mathrm{e}^{i\omega_i t}) \mathrm{Re}(B \mathrm{e}^{i\omega_j t}) = \frac{1}{2} \mathrm{Re}(A \mathrm{e}^{i\omega_i t})(B \mathrm{e}^{i\omega_j t} + B^* \mathrm{e}^{-i\omega_j t})$$

where $(^*)$ denotes the complex conjugate. Adopting a form for Q analogous to (1.3),

$$Q(\mathbf{x}, t) = \mathrm{Re} \sum_i \sum_j \left[Q_{ij}^{+}(\mathbf{x}) e^{i(\omega_i + \omega_j)t} + Q_{ij}^{-}(\mathbf{x}) e^{i(\omega_i - \omega_j)t} \right] \tag{1.8}$$

As in (1.4) it is appropriate to symmetrize the functions $Q_{ij}^{\pm}$ such that

$$Q_{ij}^{+} = Q_{ji}^{+} \quad \text{and} \quad Q_{ij}^{-} = Q_{ji}^{-*} \tag{1.9}$$

Combining (1.2) and (1.7) gives the expressions

$$\begin{aligned} Q_{ij}^{+} = & \frac{i}{4g} \omega_i \phi_i \left(-\omega_j^2 \frac{\partial \phi_j}{\partial z} + g \frac{\partial^2 \phi_j}{\partial z^2} \right) + \frac{i}{4g} \omega_j \phi_j \left(-\omega_i^2 \frac{\partial \phi_i}{\partial z} + g \frac{\partial^2 \phi_i}{\partial z^2} \right) \\ & - \frac{1}{2} i (\omega_i + \omega_j) \nabla \phi_i \cdot \nabla \phi_j \end{aligned} \tag{1.10}$$

and

$$Q_{ij}^{-} = \frac{i}{4g}\omega_i\phi_i(-\omega_j^2\frac{\partial\phi_j^*}{\partial z} + g\frac{\partial^2\phi_j^*}{\partial z^2}) - \frac{i}{4g}\omega_j\phi_j^*(-\omega_i^2\frac{\partial\phi_i}{\partial z} + g\frac{\partial^2\phi_i}{\partial z^2}) \\ - \frac{1}{2}i(\omega_i - \omega_j)\nabla\phi_i \cdot \nabla\phi_j^* \tag{1.11}$$

where the right-hand-sides of (1.10-11) are evaluated on $z = 0$. With these definitions the free-surface boundary condition for the second-order potential is given by

$$-(\omega_i \pm \omega_j)^2\phi_{ij}^{\pm} + g\frac{\partial\phi_{ij}^{\pm}}{\partial z} = Q_{ij}^{\pm} \tag{1.12}$$

on $z = 0$.

2. INCIDENT WAVES

The incident wave potential Φ_I is defined to be the total first- and second-order velocity potential that would exist in the absence of the body. Each first-order component is defined by the amplitude A_j, frequency ω_j, and vector wavenumber $\mathbf{K}_j$ with Cartesian components $(\kappa_j \cos\beta_j, \kappa_j \sin\beta_j, 0)$. Here β_j is the angle of incidence relative to the $x-$axis. The total first-order velocity potential is

$$\Phi_I^{(1)} = \mathrm{Re}\, ig \sum_j \frac{A_j}{\omega_j} Z(\kappa_j z) \exp i(\omega_j t - \mathbf{K}_j \cdot \mathbf{x}) \tag{2.1}$$

Here, for a fluid of depth h,

$$Z(\kappa_j z) = \frac{\cosh(\kappa_j (z+h))}{\cosh(\kappa_j h)} \tag{2.2}$$

In accordance with the first-order free-surface condition,

$$\kappa_j \tanh(\kappa_j h) = \omega_j^2/g \equiv \nu_j \tag{2.3}$$

It is helpful to anticipate the relations

$$\left[\frac{\partial Z(\kappa_j z)}{\partial z}\right]_{z=0} = \nu_j \quad \text{and} \quad \left[\frac{\partial^2 Z(\kappa_j z)}{\partial z^2}\right]_{z=0} = \kappa_j^2 \tag{2.4}$$

Combining (2.1) and (1.10-11) gives

$$\begin{aligned} Q_{ij}^{+} = & -\frac{1}{2} i g^2 A_i A_j \exp(-i(\mathbf{K}_i + \mathbf{K}_j)\cdot \mathbf{x}) \\ & \left[\left(\frac{\kappa_j^2 - \nu_j^2}{2\omega_j}\right) + \left(\frac{\kappa_i^2 - \nu_i^2}{2\omega_i}\right) + \frac{(\omega_i + \omega_j)}{\omega_i \omega_j}(\mathbf{K}_i \cdot \mathbf{K}_j - \nu_i \nu_j)\right] \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} Q_{ij}^{-} = & \frac{1}{2} i g^2 A_i A_j^* \exp(-i(\mathbf{K}_i - \mathbf{K}_j)\cdot \mathbf{x}) \\ & \left[\left(\frac{\kappa_j^2 - \nu_j^2}{2\omega_j}\right) - \left(\frac{\kappa_i^2 - \nu_i^2}{2\omega_i}\right) - \frac{(\omega_i - \omega_j)}{\omega_i \omega_j}\Big(\mathbf{K}_i \cdot \mathbf{K}_j + \nu_i \nu_j\Big)\right] \end{aligned} \tag{2.6}$$

Solutions for the second-order components of the incident-wave potential follow from (1.12) in the form

$$\phi_{ij}^{\pm} = \frac{Q_{ij}^{\pm}(x,y) Z(\kappa_{ij}^{\pm} z)}{-(\omega_i \pm \omega_j)^2 + g\kappa_{ij}^{\pm} \tanh \kappa_{ij}^{\pm} h} \tag{2.7}$$

where

$$\kappa_{ij}^{\pm} = |\mathbf{K}_i \pm \mathbf{K}_j| \tag{2.8}$$

Subsequently in the analysis of the free-surface integral of the second-order problem, a Fourier-Bessel expansion of the first-order incident-wave potential will be required. In terms of the polar coordinates $x = \rho \cos\theta, \quad y = \rho \sin\theta$ the appropriate expansion is

$$\phi_{Ij} = \frac{igA_j}{\omega_j} \; Z(\kappa_j z) \sum_{n=0}^{\infty} \epsilon_n (-i)^n J_n(\kappa_j \rho) \cos n(\theta - \beta_j) \tag{2.9}$$

where $\epsilon_0 = 1$, $\epsilon_n = 2$ for $(n \geq 1)$, and J_n is the first-kind Bessel function of order n. It will be convenient to redefine the coefficients in (2.9), writing this equation in the form

$$\phi_{Ij} = Z(\kappa_j z) \sum_{n=0}^{\infty} J_n(\kappa_j \rho)[A_{jn}^c \cos n\theta + A_{jn}^s \sin n\theta] \tag{2.10}$$

where the coefficients are

$$\begin{pmatrix} A_{jn}^c \\ A_{jn}^s \end{pmatrix} = \frac{igA_j}{\omega_j} \epsilon_n (-i)^n \begin{pmatrix} \cos n\beta_j \\ \sin n\beta_j \end{pmatrix} \tag{2.11}$$

3. **VELOCITY POTENTIAL DUE TO THE BODY**

In this section general properties will be listed for the solution of the velocity potential Φ_B due to the presence of the body. This includes both the scattered potential Φ_S which accounts for the effects of the incident waves on the fixed body, and for the radiation potential Φ_R due to the motions of the body about its fixed mean position. In all cases the decompositions and expansions (1.1-1.3) are applicable.

The boundary condition on the mean position $\overline{S}_B$ of the body can be written in the generic form

$$\frac{\partial \phi}{\partial n} = q_B \quad \text{on } \overline{S}_B \tag{3.1}$$

Similarly, on the free surface,

$$-\omega^2 \phi + g \frac{\partial \phi}{\partial z} = q_F \quad \text{on } z = 0 \tag{3.2}$$

where ω is the frequency corresponding to the component ϕ.

The solution for ϕ can be derived from Green's theorem in terms of the integral equation

$$2\pi\phi(\mathbf{x}) + \iint_{\overline{S}_B} \phi(\boldsymbol{\xi}) \frac{\partial G(\mathbf{x};\boldsymbol{\xi})}{\partial n_\xi} dS = \iint_{\overline{S}_B} q_B(\boldsymbol{\xi}) G(\mathbf{x};\boldsymbol{\xi}) dS + \iint_{\overline{S}_F} q_F(\boldsymbol{\xi}) G(\mathbf{x};\boldsymbol{\xi}) dS \tag{3.3}$$

In this equation ϕ represents any first- or second-order velocity potential, subject to (3.1) and (3.2) and a suitable radiation condition at infinity. G is the free-surface Green function corresponding to the same frequency ω and wavenumber as ϕ. Since (3.3) can be applied to an arbitrary closed surface, this equation holds on the mean position of S_B and on the plane $z = 0$ corresponding to the mean position of S_F, and it is not necessary to apply (3.3) to the exact boundaries of the fluid domain.

For the first-order components of the potential $q_F = 0$, and (3.3) reduces to the conventional form of the integral equation for ϕ on the body. In the fluid domain the solution for each first-order component of the potential can be expressed in the form

$$\phi(\mathbf{x}) = \iint_{\overline{S}_B} \sigma(\boldsymbol{\xi}) G(\mathbf{x};\boldsymbol{\xi}) dS \tag{3.4}$$

For the formulation based on Green's theorem, σ is the linear operator

$$\sigma = \frac{1}{4\pi}\Big(q_B - \phi\frac{\partial}{\partial n}\Big) \tag{3.5}$$

Alternatively, in the source formulation, the parameter σ is defined as the first-order source strength, and can be found directly as the solution of the integral equation

$$2\pi\sigma(\mathbf{x}) + \iint_{\overline{S}_B} \sigma(\boldsymbol{\xi}) \frac{\partial G(\mathbf{x};\boldsymbol{\xi})}{\partial n_x} dS = q_B(x) \tag{3.6}$$

In the subsequent analysis of the free-surface integral of the second-order problem, a Fourier expansion of the first-order body potential will be required to complement (2.9) in the far-field domain outside of a partition circle of radius b which is sufficiently large to completely surround the body. Consider a generic wavenumber κ which may be equal to either κ_i, κ_j, or the wavenumber κ_k of the second-order potential. With the additional restriction that κb is sufficiently large, evanascent (local) nonradiating components of the Green function can be excluded from consideration, and

$$G \simeq 2\pi i c Z(\kappa z) Z(\kappa\zeta) H_0^{(2)}(\kappa R) \tag{3.7}$$

Here the function Z is defined by (2.2), R is the horizontal distance between the source and field points $(\mathbf{x}, \boldsymbol{\xi})$, and $H_0^{(2)}(\kappa R) = J_0(\kappa R) - iY_0(\kappa R)$ is the Hankel function of the second kind. (In the following analysis the superscript (2) will be deleted.) The constant c is defined by

$$c = \frac{\nu^2 - \kappa^2}{\kappa^2 h - \nu^2 h + \nu} \cosh^2(\kappa h) = \frac{-\kappa^2}{\kappa^2 h - \nu^2 h + \nu} \tag{3.8}$$

The error in the far-field approximation (3.7) is of order $\exp(-CR/h)$ where the constant C is greater than $\pi/2$. In the infinite-depth limit the corresponding error is of order $(\kappa R)^{-2}$, or of order $(\kappa R)^{-3}$ in the special case where $z = 0$.

Graf's addition theorem (Abramowitz & Stegun, eq. 9.1.79) may be used to expand (3.7) in a Fourier-Bessel series analogous to (2.9):

$$G(\mathbf{x}; \boldsymbol{\xi}) \simeq 2\pi i c Z(\kappa z) Z(\kappa\zeta) \sum_{n=0}^{\infty} \epsilon_n H_n(\kappa\rho) J_n(\kappa\rho') \cos n(\theta - \theta') \tag{3.9}$$

where $\xi = \rho' \cos\theta'$, $\eta = \rho' \sin\theta'$, and (3.9) is valid in the domain $(\rho \geq \rho')$. Substituting this result into (3.4) gives the corresponding far-field representation of the body potential

$$\phi(\mathbf{x}) \simeq Z(\kappa z) \sum_{n=0}^{\infty} H_n(\kappa\rho) [B_n^c \cos n\theta + B_n^s \sin n\theta] \tag{3.10}$$

where the coefficients in this expansion are

$$\begin{pmatrix} B_n^c \\ B_n^s \end{pmatrix} = 2\pi i c \epsilon_n \iint_{\overline{S}_B} \sigma(\boldsymbol{\xi}) Z(\kappa\zeta) J_n(\kappa\rho') \begin{pmatrix} \cos n\theta' \\ \sin n\theta' \end{pmatrix} dS \tag{3.11}$$

4. THE FREE-SURFACE INTEGRAL

The free-surface integral which is displayed as the last term in (3.3) will be considered here, with the 'quadratic forcing function' q_F defined by the relations (1.6-12). Since we are considering the evaluation of the second-order velocity potential Φ_B due to the presence of the body, the incident-wave potential Φ_I should be subtracted from the left side of the free-surface conditions (1.6) and (1.12), and also from the evaluation of the quadratic forcing function. Thus we seek to evaluate the difference between (1.10-1.11) and (2.5-2.6), which includes terms on the right side of (1.10-1.11) due to (a) quadratic interactions of the body potential with itself, and (b) cross-terms between Φ_B and Φ_I.

For purposes of numerical evaluation, the free-surface integral is composed of two parts, separated by the partition circle of radius $\rho = b$. Following the methodology of Kim and Yue, the integration in the inner domain $\rho < b$ is carried out numerically. Thus the inner free surface is discretized with quadrilateral panels, in an analogous manner to the body surface. To avoid the evaluation of second-order derivatives of the first-order potentials in (1.10-11), the surface integral is transformed using Stokes' theorem to a form involving only first-order derivatives and line integrals around the inner and outer boundaries. Details are given in the paper by Kim.

4.1. THE FREE-SURFACE INTEGRAL IN THE FAR FIELD

In the remainder of this section the 'far-field integral' in the domain ($b \le \rho < \infty$) is considered. For this purpose the incident- and body-potentials are defined by the Fourier-Bessel expansions (2.10) and (3.10), respectively.

The first task is to evaluate (1.10) and (1.11). We begin with the sum-frequency component Q^{+}_{ij}, considering first the 'BB' component due to quadratic interactions of the body potential (3.10). Associating the Fourier indices (ℓ, m) with the frequency indices (i, j) respectively, we obtain

$$\begin{aligned} Q^{+}_{BiBj} = \frac{i}{4}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\Big\{ & \left[\omega_i(\kappa_j^2-\nu_j^2)+\omega_j(\kappa_i^2-\nu_i^2)-2(\omega_i+\omega_j)\nu_i\nu_j\right] \\ & H_\ell(\kappa_i\rho)H_m(\kappa_j\rho) \\ & \qquad (B^c_{i\ell}\cos\ell\theta + B^s_{i\ell}\sin\ell\theta)(B^c_{jm}\cos m\theta + B^s_{jm}\sin m\theta) \\ & -2(\omega_i+\omega_j)\kappa_i\kappa_j H'_\ell(\kappa_i\rho)H'_m(\kappa_j\rho) \\ & \qquad (B^c_{i\ell}\cos\ell\theta + B^s_{i\ell}\sin\ell\theta)(B^c_{jm}\cos m\theta + B^s_{jm}\sin m\theta) \\ & -2(\omega_i+\omega_j)\frac{\ell m}{\rho^2}H_\ell(\kappa_i\rho)H_m(\kappa_j\rho) \\ & \qquad (B^s_{i\ell}\cos\ell\theta - B^c_{i\ell}\sin\ell\theta)(B^s_{jm}\cos m\theta - B^c_{jm}\sin m\theta)\Big\} \end{aligned} \tag{4.1}$$

where H' denotes the first derivative of the Hankel function with respect to its argument.

4.2. THE FOURIER INTEGRALS IN THE ANGULAR COORDINATE

The integration with respect to the angular coordinate θ can be evaluated after multiplication by the angular components of the Green function (3.9). Thus we consider the Fourier integrals

$$\int_0^{2\pi} Q^{+}_{ij}\begin{pmatrix}\cos n\theta\\ \sin n\theta\end{pmatrix} d\theta$$

The only nonzero components involve the integrals

$$\int_0^{2\pi}\cos\ell\theta\cos m\theta\cos n\theta d\theta = \frac{\pi}{\epsilon_n}[\delta_{n,|\ell-m|}+\delta_{n,\ell+m}] \equiv \lambda^{+}_{\ell mn} \tag{4.2a}$$

$$\int_0^{2\pi}\sin\ell\theta\sin m\theta\cos n\theta d\theta = \frac{\pi}{\epsilon_n}[\delta_{n,|\ell-m|}-\delta_{n,\ell+m}] \equiv \lambda^{-}_{\ell mn} \tag{4.2b}$$

$$\int_0^{2\pi} \cos \ell\theta \sin m\theta \sin n\theta d\theta = \frac{\pi}{\epsilon_\ell}[\delta_{\ell,|n-m|} - \delta_{\ell,n+m}] = \lambda^-_{mn\ell} \tag{4.2c}$$

$$\int_0^{2\pi} \sin \ell\theta \cos m\theta \sin n\theta d\theta = \frac{\pi}{\epsilon_m}[\delta_{m,|n-\ell|} - \delta_{m,n+\ell}] = \lambda^-_{n\ell m} \tag{4.2d}$$

where δ_{mn} is the Kronecker delta function, equal to one if the subscripts are equal and otherwise equal to zero. The factors λ defined above are nonzero if and only if $\ell = m + n$, or $m = n + \ell$, or $n = \ell + m$.

The Fourier integrals of the quadratic forcing function (4.1) are then given by

$$\begin{aligned}\int_0^{2\pi} Q^+_{BiBj} \begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix} d\theta = \frac{i}{4} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \Big\{ \big[\omega_i(\kappa_j^2 - \nu_j^2) + \omega_j(\kappa_i^2 - \nu_i^2) - 2(\omega_i + \omega_j)\nu_i\nu_j\big] \\ H_\ell(\kappa_i\rho)H_m(\kappa_j\rho) \begin{pmatrix} B^c_{i\ell}B^c_{jm}\lambda^+_{\ell mn} + B^s_{i\ell}B^s_{jm}\lambda^-_{\ell mn} \\ B^s_{i\ell}B^c_{jm}\lambda^-_{n\ell m} + B^c_{i\ell}B^s_{jm}\lambda^-_{mn\ell} \end{pmatrix} \\ - 2(\omega_i + \omega_j)\Big[\kappa_i\kappa_j H'_\ell(\kappa_i\rho)H'_m(\kappa_j\rho) \begin{pmatrix} B^c_{i\ell}B^c_{jm}\lambda^+_{\ell mn} + B^s_{i\ell}B^s_{jm}\lambda^-_{\ell mn} \\ B^s_{i\ell}B^c_{jm}\lambda^-_{n\ell m} + B^c_{i\ell}B^s_{jm}\lambda^-_{mn\ell} \end{pmatrix} \\ + \frac{\ell m}{\rho^2} H_\ell(\kappa_i\rho)H_m(\kappa_j\rho) \begin{pmatrix} B^c_{i\ell}B^c_{jm}\lambda^-_{\ell mn} + B^s_{i\ell}B^s_{jm}\lambda^+_{\ell mn} \\ -B^c_{i\ell}B^s_{jm}\lambda^-_{n\ell m} - B^s_{i\ell}B^c_{jm}\lambda^-_{mn\ell} \end{pmatrix}\Big]\Big\}\end{aligned} \tag{4.3}$$

The second term, which involves derivatives of the Hankel functions, can be reduced using the relation $H'_\nu = \frac{1}{2}(H_{\nu-1} - H_{\nu+1})$. Thus

$$H'_\ell(\kappa_i\rho)H'_m(\kappa_j\rho) = \frac{1}{4}\Big(H_{\ell-1}H_{m-1} - H_{\ell-1}H_{m+1} + H_{\ell+1}H_{m+1} - H_{\ell+1}H_{m-1}\Big) \tag{4.4}$$

Similarly, the third term resulting from the angular derivatives can be reduced using the recurrence relation $2\nu H_\nu/z = H_{\nu-1} + H_{\nu+1}$ to remove the factor ρ^{-2}. Three alternative relations result; the one which is most useful is obtained by setting ν equal to ℓ and then to m:

$$H_\ell(\kappa_i\rho)H_m(\kappa_j\rho)\rho^{-2} = \frac{\kappa_i\kappa_j}{4\ell m}\Big(H_{\ell-1}H_{m-1} + H_{\ell-1}H_{m+1} + H_{\ell+1}H_{m+1} + H_{\ell+1}H_{m-1}\Big) \tag{4.5}$$

Substituting (4.4-5) in (4.3) gives

$$\int_0^{2\pi} Q^+_{BiBj} \binom{\cos n\theta}{\sin n\theta} d\theta = \frac{i}{4} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \Big\{ \big[\omega_i(\kappa_j^2 - \nu_j^2) + \omega_j(\kappa_i^2 - \nu_i^2) - 2(\omega_i + \omega_j)\nu_i\nu_j\big]$$
$$H_\ell(\kappa_i\rho)H_m(\kappa_j\rho) \begin{pmatrix} B^c_{i\ell}B^c_{jm}\lambda^+_{\ell mn} + B^s_{i\ell}B^s_{jm}\lambda^-_{\ell mn} \\ B^s_{i\ell}B^c_{jm}\lambda^-_{n\ell m} + B^c_{i\ell}B^s_{jm}\lambda^-_{mn\ell} \end{pmatrix}$$
$$- \frac{1}{2}(\omega_i + \omega_j)\kappa_i\kappa_j \Big[H_{\ell-1}(\kappa_i\rho)H_{m-1}(\kappa_j\rho) + H_{\ell+1}(\kappa_i\rho)H_{m+1}(\kappa_j\rho)\Big]$$
$$\begin{pmatrix} (B^c_{i\ell}B^c_{jm} + B^s_{i\ell}B^s_{jm})(\lambda^+_{\ell mn} + \lambda^-_{\ell mn}) \\ (B^s_{i\ell}B^c_{jm} - B^c_{i\ell}B^s_{jm})(\lambda^-_{n\ell m} - \lambda^-_{mn\ell}) \end{pmatrix} \tag{4.6}$$
$$+ \frac{1}{2}(\omega_i + \omega_j)\kappa_i\kappa_j \Big[H_{\ell-1}(\kappa_i\rho)H_{m+1}(\kappa_j\rho) + H_{\ell+1}(\kappa_i\rho)H_{m-1}(\kappa_j\rho)\Big]$$
$$\begin{pmatrix} (B^c_{i\ell}B^c_{jm} - B^s_{i\ell}B^s_{jm})(\lambda^+_{\ell mn} - \lambda^-_{\ell mn}) \\ (B^s_{i\ell}B^c_{jm} + B^c_{i\ell}B^s_{jm})(\lambda^-_{n\ell m} + \lambda^-_{mn\ell}) \end{pmatrix} \Big\}$$

The corresponding integrals for the difference-frequency function are

$$\int_0^{2\pi} Q^-_{BiBj} \binom{\cos n\theta}{\sin n\theta} d\theta = \frac{i}{4} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \Big\{ \big[\omega_i(\kappa_j^2 - \nu_j^2) - \omega_j(\kappa_i^2 - \nu_i^2) - 2(\omega_i - \omega_j)\nu_i\nu_j\big]$$
$$H_\ell(\kappa_i\rho)H^*_m(\kappa_j\rho) \begin{pmatrix} B^c_{i\ell}B^{c*}_{jm}\lambda^+_{\ell mn} + B^s_{i\ell}B^{s*}_{jm}\lambda^-_{\ell mn} \\ B^s_{i\ell}B^{c*}_{jm}\lambda^-_{n\ell m} + B^c_{i\ell}B^{s*}_{jm}\lambda^-_{mn\ell} \end{pmatrix}$$
$$- \frac{1}{2}(\omega_i - \omega_j)\kappa_i\kappa_j \Big[H_{\ell-1}(\kappa_i\rho)H^*_{m-1}(\kappa_j\rho) + H_{\ell+1}(\kappa_i\rho)H^*_{m+1}(\kappa_j\rho)\Big]$$
$$\begin{pmatrix} (B^c_{i\ell}B^{c*}_{jm} + B^s_{i\ell}B^{s*}_{jm})(\lambda^+_{\ell mn} + \lambda^-_{\ell mn}) \\ (B^s_{i\ell}B^{c*}_{jm} - B^c_{i\ell}B^{s*}_{jm})(\lambda^-_{n\ell m} - \lambda^-_{mn\ell}) \end{pmatrix} \tag{4.7}$$
$$+ \frac{1}{2}(\omega_i - \omega_j)\kappa_i\kappa_j \Big[H_{\ell-1}(\kappa_i\rho)H^*_{m+1}(\kappa_j\rho) + H_{\ell+1}(\kappa_i\rho)H^*_{m-1}(\kappa_j\rho)\Big]$$
$$\begin{pmatrix} (B^c_{i\ell}B^{c*}_{jm} - B^s_{i\ell}B^{s*}_{jm})(\lambda^+_{\ell mn} - \lambda^-_{\ell mn}) \\ (B^s_{i\ell}B^{c*}_{jm} + B^c_{i\ell}B^{s*}_{jm})(\lambda^-_{n\ell m} + \lambda^-_{mn\ell}) \end{pmatrix} \Big\}$$

It is helpful to define the factors

$$\Omega^\pm_{ij} = \omega_i(\kappa_j^2 - \nu_j^2) \pm \omega_j(\kappa_i^2 - \nu_i^2) - 2(\omega_i \pm \omega_j)\nu_i\nu_j \tag{4.8}$$

and

$$\Lambda_{ij}^{\pm} = (\omega_i \pm \omega_j)\kappa_i\kappa_j \tag{4.9}$$

(These definitions differ by multiplicative factors from the corresponding parameters used by Kim.)

Equations (4.6) and (4.7) can then be written in a more compact form, after noting that the first factor contained within large parentheses in each equation is equivalent to half the sum of the second and third factors. Thus

$$\begin{aligned}
\int_0^{2\pi} Q_{BiBj}^{+} \begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix} d\theta = \frac{i}{8} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \Big\{ \Big[& \Omega_{ij}^{+} H_\ell(\kappa_i\rho) H_m(\kappa_j\rho) \\
& - \Lambda_{ij}^{+} \Big(H_{\ell-1}(\kappa_i\rho) H_{m-1}(\kappa_j\rho) + H_{\ell+1}(\kappa_i\rho) H_{m+1}(\kappa_j\rho) \Big) \Big] \\
& \begin{pmatrix} (B_{i\ell}^c B_{jm}^c + B_{i\ell}^s B_{jm}^s)(\lambda_{\ell mn}^{+} + \lambda_{\ell mn}^{-}) \\ (B_{i\ell}^s B_{jm}^c - B_{i\ell}^c B_{jm}^s)(\lambda_{n\ell m}^{-} - \lambda_{mn\ell}^{-}) \end{pmatrix} \\
+ \Big[\Omega_{ij}^{+} H_\ell(\kappa_i\rho) H_m(\kappa_j\rho) & + \Lambda_{ij}^{+} \Big(H_{\ell-1}(\kappa_i\rho) H_{m+1}(\kappa_j\rho) + H_{\ell+1}(\kappa_i\rho) H_{m-1}(\kappa_j\rho) \Big) \Big] \\
& \begin{pmatrix} (B_{i\ell}^c B_{jm}^c - B_{i\ell}^s B_{jm}^s)(\lambda_{\ell mn}^{+} - \lambda_{\ell mn}^{-}) \\ (B_{i\ell}^s B_{jm}^c + B_{i\ell}^c B_{jm}^s)(\lambda_{n\ell m}^{-} + \lambda_{mn\ell}^{-}) \end{pmatrix} \Big\}
\end{aligned} \tag{4.10}$$

Similarly for the difference-frequency integral

$$\begin{aligned}
\int_0^{2\pi} Q_{BiBj}^{-} \begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix} d\theta = \frac{i}{8} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \Big\{ \Big[& \Omega_{ij}^{-} H_\ell(\kappa_i\rho) H_m^*(\kappa_j\rho) \\
& - \Lambda_{ij}^{-} \Big(H_{\ell-1}(\kappa_i\rho) H_{m-1}^*(\kappa_j\rho) + H_{\ell+1}(\kappa_i\rho) H_{m+1}^*(\kappa_j\rho) \Big) \Big] \\
& \begin{pmatrix} (B_{i\ell}^c B_{jm}^{c*} + B_{i\ell}^s B_{jm}^{s*})(\lambda_{\ell mn}^{+} + \lambda_{\ell mn}^{-}) \\ (B_{i\ell}^s B_{jm}^{c*} - B_{i\ell}^c B_{jm}^{s*})(\lambda_{n\ell m}^{-} - \lambda_{mn\ell}^{-}) \end{pmatrix} \\
+ \Big[\Omega_{ij}^{-} H_\ell(\kappa_i\rho) H_m^*(\kappa_j\rho) & + \Lambda_{ij}^{-} \Big(H_{\ell-1}(\kappa_i\rho) H_{m+1}^*(\kappa_j\rho) + H_{\ell+1}(\kappa_i\rho) H_{m-1}^*(\kappa_j\rho) \Big) \Big] \\
& \begin{pmatrix} (B_{i\ell}^c B_{jm}^{c*} - B_{i\ell}^s B_{jm}^{s*})(\lambda_{\ell mn}^{+} - \lambda_{\ell mn}^{-}) \\ (B_{i\ell}^s B_{jm}^{c*} + B_{i\ell}^c B_{jm}^{s*})(\lambda_{n\ell m}^{-} + \lambda_{mn\ell}^{-}) \end{pmatrix} \Big\}
\end{aligned} \tag{4.11}$$

4.3. THE INTEGRALS IN THE RADIAL COORDINATE

In the context of the free-surface integral represented by the last term in (3.3) it is necessary to multiply the preceding results by the Hankel functions in (3.9), and integrate in the radial domain $b < \rho < \infty$. Since the differential element of surface area is $\rho d\rho d\theta$, the integrals which must be evaluated in (4.10) and (4.11) are of the basic form

$$\int_b^\infty H_\ell(\kappa_i\rho)H_m(\kappa_j\rho)H_n(\kappa_k\rho)\rho d\rho = b^2\int_1^\infty H_\ell(\alpha x)H_m(\beta x)H_n(\gamma x)x dx \equiv b^2\mathcal{F}_{\ell mn}(\alpha,\beta,\gamma) \tag{4.12}$$

Additional integrals will also be required to analyze the interaction between the body and incident-wave potentials, where one of the first pair of Hankel functions in (4.12) is replaced by the corresponding Bessel function J_ν, of the same order and with the same argument. Similarly, in (4.11), the second Hankel function is replaced by its complex conjugate or the Hankel function of the first kind.

To make the above results more explicit, we define the following nondimensional integrals:

$$\mathcal{F}^{(1,2)}_{\ell mn} = \int_1^\infty H^{(2)}_\ell(\alpha x)H^{(1,2)}_m(\beta x)H^{(2)}_n(\gamma x)x dx \tag{4.13a}$$

$$\mathcal{G}^{(1,2)}_{\ell mn} = \int_1^\infty J_\ell(\alpha x)H^{(1,2)}_m(\beta x)H^{(2)}_n(\gamma x)x dx \tag{4.13b}$$

$$\mathcal{H}^{(1,2)}_{\ell mn} = \int_1^\infty H^{(2)}_\ell(\alpha x)J_m(\beta x)H^{(2)}_n(\gamma x)x dx \tag{4.13c}$$

(The superscript for the third function is superfluous, but it is retained for uniformity in the notation.) With these definitions applied, and the contributions included from the incident-wave potential, the appropriate radial integrals are

$$
\begin{aligned}
&\int_b^\infty H_n^{(2)}(\kappa_k\rho)\rho d\rho \int_0^{2\pi} Q_{ij}^+(\rho,\theta)\binom{\cos n\theta}{\sin n\theta} d\theta = \\
&\quad \frac{i}{8}b^2 \sum_{\ell=0}^\infty \sum_{m=0}^\infty \Big\{ \Big[\Omega_{ij}^+ \mathcal{F}_{\ell,m,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b) - \Lambda_{ij}^+\big(\mathcal{F}_{\ell-1,m-1,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b) \\
&\qquad\quad + \mathcal{F}_{\ell+1,m+1,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b)\big)\Big] \begin{pmatrix} (B_{i\ell}^c B_{jm}^c + B_{i\ell}^s B_{jm}^s)(\lambda_{\ell mn}^+ + \lambda_{\ell mn}^-) \\ (B_{i\ell}^s B_{jm}^c - B_{i\ell}^c B_{jm}^s)(\lambda_{n\ell m}^- - \lambda_{mn\ell}^-) \end{pmatrix} \\
&\quad + \Big[\Omega_{ij}^+ \mathcal{F}_{\ell,m,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b) + \Lambda_{ij}^+\big(\mathcal{F}_{\ell-1,m+1,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b) \\
&\qquad\quad + \mathcal{F}_{\ell+1,m-1,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b)\big)\Big] \begin{pmatrix} (B_{i\ell}^c B_{jm}^c - B_{i\ell}^s B_{jm}^s)(\lambda_{\ell mn}^+ - \lambda_{\ell mn}^-) \\ (B_{i\ell}^s B_{jm}^c + B_{i\ell}^c B_{jm}^s)(\lambda_{n\ell m}^- + \lambda_{mn\ell}^-) \end{pmatrix} \\
&\quad + \Big[\Omega_{ij}^+ \mathcal{G}_{\ell,m,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b) - \Lambda_{ij}^+\big(\mathcal{G}_{\ell-1,m-1,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b) \\
&\qquad\quad + \mathcal{G}_{\ell+1,m+1,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b)\big)\Big] \begin{pmatrix} (A_{i\ell}^c B_{jm}^c + A_{i\ell}^s B_{jm}^s)(\lambda_{\ell mn}^+ + \lambda_{\ell mn}^-) \\ (A_{i\ell}^s B_{jm}^c - A_{i\ell}^c B_{jm}^s)(\lambda_{n\ell m}^- - \lambda_{mn\ell}^-) \end{pmatrix} \\
&\quad + \Big[\Omega_{ij}^+ \mathcal{G}_{\ell,m,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b) + \Lambda_{ij}^+\big(\mathcal{G}_{\ell-1,m+1,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b) \\
&\qquad\quad + \mathcal{G}_{\ell+1,m-1,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b)\big)\Big] \begin{pmatrix} (A_{i\ell}^c B_{jm}^c - A_{i\ell}^s B_{jm}^s)(\lambda_{\ell mn}^+ - \lambda_{\ell mn}^-) \\ (A_{i\ell}^s B_{jm}^c + A_{i\ell}^c B_{jm}^s)(\lambda_{n\ell m}^- + \lambda_{mn\ell}^-) \end{pmatrix} \\
&\quad + \Big[\Omega_{ij}^+ \mathcal{H}_{\ell,m,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b) - \Lambda_{ij}^+\big(\mathcal{H}_{\ell-1,m-1,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b) \\
&\qquad\quad + \mathcal{H}_{\ell+1,m+1,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b)\big)\Big] \begin{pmatrix} (B_{i\ell}^c A_{jm}^c + B_{i\ell}^s A_{jm}^s)(\lambda_{\ell mn}^+ + \lambda_{\ell mn}^-) \\ (B_{i\ell}^s A_{jm}^c - B_{i\ell}^c A_{jm}^s)(\lambda_{n\ell m}^- - \lambda_{mn\ell}^-) \end{pmatrix} \\
&\quad + \Big[\Omega_{ij}^+ \mathcal{H}_{\ell,m,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b) + \Lambda_{ij}^+\big(\mathcal{H}_{\ell-1,m+1,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b) \\
&\qquad\quad + \mathcal{H}_{\ell+1,m-1,n}^{(2)}(\kappa_i b,\kappa_j b,\kappa_k b)\big)\Big] \begin{pmatrix} (B_{i\ell}^c A_{jm}^c - B_{i\ell}^s A_{jm}^s)(\lambda_{\ell mn}^+ - \lambda_{\ell mn}^-) \\ (B_{i\ell}^s A_{jm}^c + B_{i\ell}^c A_{jm}^s)(\lambda_{n\ell m}^- + \lambda_{mn\ell}^-) \end{pmatrix} \Big\}
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
&\int_b^\infty H_n^{(2)}(\kappa_k\rho)\rho d\rho \int_0^{2\pi} Q_{ij}^-(\rho,\theta)\begin{pmatrix}\cos n\theta\\ \sin n\theta\end{pmatrix} d\theta = \\
&\quad \frac{i}{8}b^2 \sum_{\ell=0}^\infty \sum_{m=0}^\infty \Big\{ \Big[\Omega_{ij}^- \mathcal{F}^{(1)}_{\ell,m,n}(\kappa_i b,\kappa_j b,\kappa_k b) - \Lambda_{ij}^-\big(\mathcal{F}^{(1)}_{\ell-1,m-1,n}(\kappa_i b,\kappa_j b,\kappa_k b) \\
&\qquad\quad + \mathcal{F}^{(1)}_{\ell+1,m+1,n}(\kappa_i b,\kappa_j b,\kappa_k b)\big)\Big]\begin{pmatrix}(B_{i\ell}^c B_{jm}^{*c} + B_{i\ell}^s B_{jm}^{*s})(\lambda_{\ell mn}^+ + \lambda_{\ell mn}^-)\\ (B_{i\ell}^s B_{jm}^{*c} - B_{i\ell}^c B_{jm}^{*s})(\lambda_{n\ell m}^- - \lambda_{mn\ell}^-)\end{pmatrix} \\
&\quad + \Big[\Omega_{ij}^- \mathcal{F}^{(1)}_{\ell,m,n}(\kappa_i b,\kappa_j b,\kappa_k b) + \Lambda_{ij}^-\big(\mathcal{F}^{(1)}_{\ell-1,m+1,n}(\kappa_i b,\kappa_j b,\kappa_k b) \\
&\qquad\quad + \mathcal{F}^{(1)}_{\ell+1,m-1,n}(\kappa_i b,\kappa_j b,\kappa_k b)\big)\Big]\begin{pmatrix}(B_{i\ell}^c B_{jm}^{*c} - B_{i\ell}^s B_{jm}^{*s})(\lambda_{\ell mn}^+ - \lambda_{\ell mn}^-)\\ (B_{i\ell}^s B_{jm}^{*c} + B_{i\ell}^c B_{jm}^{*s})(\lambda_{n\ell m}^- + \lambda_{mn\ell}^-)\end{pmatrix} \\
&\quad + \Big[\Omega_{ij}^- \mathcal{G}^{(1)}_{\ell,m,n}(\kappa_i b,\kappa_j b,\kappa_k b) - \Lambda_{ij}^-\big(\mathcal{G}^{(1)}_{\ell-1,m-1,n}(\kappa_i b,\kappa_j b,\kappa_k b) \\
&\qquad\quad + \mathcal{G}^{(1)}_{\ell+1,m+1,n}(\kappa_i b,\kappa_j b,\kappa_k b)\big)\Big]\begin{pmatrix}(A_{i\ell}^c B_{jm}^{*c} + A_{i\ell}^s B_{jm}^{*s})(\lambda_{\ell mn}^+ + \lambda_{\ell mn}^-)\\ (A_{i\ell}^s B_{jm}^{*c} - A_{i\ell}^c B_{jm}^{*s})(\lambda_{n\ell m}^- - \lambda_{mn\ell}^-)\end{pmatrix} \\
&\quad + \Big[\Omega_{ij}^- \mathcal{G}^{(1)}_{\ell,m,n}(\kappa_i b,\kappa_j b,\kappa_k b) + \Lambda_{ij}^-\big(\mathcal{G}^{(1)}_{\ell-1,m+1,n}(\kappa_i b,\kappa_j b,\kappa_k b) \\
&\qquad\quad + \mathcal{G}^{(1)}_{\ell+1,m-1,n}(\kappa_i b,\kappa_j b,\kappa_k b)\big)\Big]\begin{pmatrix}(A_{i\ell}^c B_{jm}^{*c} - A_{i\ell}^s B_{jm}^{*s})(\lambda_{\ell mn}^+ - \lambda_{\ell mn}^-)\\ (A_{i\ell}^s B_{jm}^{*c} + A_{i\ell}^c B_{jm}^{*s})(\lambda_{n\ell m}^- + \lambda_{mn\ell}^-)\end{pmatrix} \\
&\quad + \Big[\Omega_{ij}^- \mathcal{H}^{(1)}_{\ell,m,n}(\kappa_i b,\kappa_j b,\kappa_k b) - \Lambda_{ij}^-\big(\mathcal{H}^{(1)}_{\ell-1,m-1,n}(\kappa_i b,\kappa_j b,\kappa_k b) \\
&\qquad\quad + \mathcal{H}^{(1)}_{\ell+1,m+1,n}(\kappa_i b,\kappa_j b,\kappa_k b)\big)\Big]\begin{pmatrix}(B_{i\ell}^c A_{jm}^{*c} + B_{i\ell}^s A_{jm}^{*s})(\lambda_{\ell mn}^+ + \lambda_{\ell mn}^-)\\ (B_{i\ell}^s A_{jm}^{*c} - B_{i\ell}^c A_{jm}^{*s})(\lambda_{n\ell m}^- - \lambda_{mn\ell}^-)\end{pmatrix} \\
&\quad + \Big[\Omega_{ij}^- \mathcal{H}^{(1)}_{\ell,m,n}(\kappa_i b,\kappa_j b,\kappa_k b) + \Lambda_{ij}^-\big(\mathcal{H}^{(1)}_{\ell-1,m+1,n}(\kappa_i b,\kappa_j b,\kappa_k b) \\
&\qquad\quad + \mathcal{H}^{(1)}_{\ell+1,m-1,n}(\kappa_i b,\kappa_j b,\kappa_k b)\big)\Big]\begin{pmatrix}(B_{i\ell}^c A_{jm}^{*c} - B_{i\ell}^s A_{jm}^{*s})(\lambda_{\ell mn}^+ - \lambda_{\ell mn}^-)\\ (B_{i\ell}^s A_{jm}^{*c} + B_{i\ell}^c A_{jm}^{*s})(\lambda_{n\ell m}^- + \lambda_{mn\ell}^-)\end{pmatrix}\Big\}
\end{aligned}
\tag{4.15}
$$

The four factors which involve sums and differences of the parameters λ will now be considered, with reference to the definitions (4.2). Excluding the special case $\ell = m = n = 0$, where

$$\lambda^+_{000} = 2\pi \quad \text{and} \quad \lambda^-_{000} = 0$$

the four factors can be simplified in the forms

$$(\lambda^+_{\ell mn} + \lambda^-_{\ell mn}) = \pi\delta_{\ell,m+n} + \pi\delta_{m,\ell+n} \tag{4.16a}$$

$$(\lambda^-_{n\ell m} - \lambda^-_{mn\ell}) = \pi\delta_{\ell,m+n} - \pi\delta_{m,\ell+n} \tag{4.16b}$$

$$(\lambda^+_{\ell mn} - \lambda^-_{\ell mn}) = \pi\delta_{n,\ell+m} \tag{4.16c}$$

$$(\lambda^-_{n\ell m} + \lambda^-_{mn\ell}) = \pi\delta_{n,\ell+m} \tag{4.16d}$$

In the three-dimensional space ℓ, m, n, the first two factors are nonzero only on the two planes $\ell = m + n$ and $m = \ell + n$, whereas the last two factors are nonzero only on the complementary plane $n = \ell + m$. Collectively these three planes form a cone with triangular sections, and with its apex at the origin. The four expressions (4.16) apply also on the intersections of each pair of adjacent planes. Thus the double sums in (4.14-15) can be expressed in terms of the separate contributions from each of the three surfaces. A compact form can be derived if we first define the three functionals

$$U^{(ij)\pm}_{\ell m}(\mathcal{F}) = \Omega^\pm_{ij}\mathcal{F}_{\ell,m,\ell-m}(\kappa_i b, \kappa_j b, \kappa_k b) - \Lambda^\pm_{ij}\big[\mathcal{F}_{\ell-1,m-1,\ell-m}(\kappa_i b, \kappa_j b, \kappa_k b) + \mathcal{F}_{\ell+1,m+1,\ell-m}(\kappa_i b, \kappa_j b, \kappa_k b)\big] \tag{4.17a}$$

$$V^{(ij)\pm}_{\ell m}(\mathcal{F}) = \Omega^\pm_{ij}\mathcal{F}_{\ell,m,m-\ell}(\kappa_i b, \kappa_j b, \kappa_k b) - \Lambda^\pm_{ij}\big[\mathcal{F}_{\ell-1,m-1,m-\ell}(\kappa_i b, \kappa_j b, \kappa_k b) + \mathcal{F}_{\ell+1,m+1,m-\ell}(\kappa_i b, \kappa_j b, \kappa_k b)\big] \tag{4.17b}$$

$$W^{(ij)\pm}_{\ell m}(\mathcal{F}) = \Omega^\pm_{ij}\mathcal{F}_{\ell,m,\ell+m}(\kappa_i b, \kappa_j b, \kappa_k b) + \Lambda^\pm_{ij}\big[\mathcal{F}_{\ell-1,m+1,\ell+m}(\kappa_i b, \kappa_j b, \kappa_k b) + \mathcal{F}_{\ell+1,m-1,\ell+m}(\kappa_i b, \kappa_j b, \kappa_k b)\big] \tag{4.17c}$$

If the series are truncated at the upper limit $\ell = M$, $m = M$, (4.14-15) are equivalent to

$$
\begin{aligned}
&\int_b^\infty H_n^{(2)}(\kappa_k\rho)\rho d\rho \int_0^{2\pi} Q_{ij}^{+}(\rho,\theta)\begin{pmatrix}\cos n\theta\\ \sin n\theta\end{pmatrix} d\theta = \\
&\qquad \frac{\pi i}{8} b^2 \Big\{ \sum_{\ell=n}^{M} (1+\delta_{\ell n}) \Big[U_{\ell m}^{(ij)+}(\mathcal{F}^{(2)}) \begin{pmatrix} (B_{i\ell}^c B_{jm}^c + B_{i\ell}^s B_{jm}^s) \\ (B_{i\ell}^s B_{jm}^c - B_{i\ell}^c B_{jm}^s) \end{pmatrix} \\
&\qquad\qquad + U_{\ell m}^{(ij)+}(\mathcal{G}^{(2)}) \begin{pmatrix} (A_{i\ell}^c B_{jm}^c + A_{i\ell}^s B_{jm}^s) \\ (A_{i\ell}^s B_{jm}^c - A_{i\ell}^c B_{jm}^s) \end{pmatrix} \\
&\qquad\qquad + U_{\ell m}^{(ij)+}(\mathcal{H}^{(2)}) \begin{pmatrix} (B_{i\ell}^c A_{jm}^c + B_{i\ell}^s A_{jm}^s) \\ (B_{i\ell}^s A_{jm}^c - B_{i\ell}^c A_{jm}^s) \end{pmatrix} \Big]_{m=\ell-n} \\
&\qquad \pm \sum_{\ell=0}^{M-n} (1+\delta_{\ell 0}) \Big[V_{\ell m}^{(ij)+}(\mathcal{F}^{(2)}) \begin{pmatrix} (B_{i\ell}^c B_{jm}^c + B_{i\ell}^s B_{jm}^s) \\ (B_{i\ell}^s B_{jm}^c - B_{i\ell}^c B_{jm}^s) \end{pmatrix} \\
&\qquad\qquad + V_{\ell m}^{(ij)+}(\mathcal{G}^{(2)}) \begin{pmatrix} (A_{i\ell}^c B_{jm}^c + A_{i\ell}^s B_{jm}^s) \\ (A_{i\ell}^s B_{jm}^c - A_{i\ell}^c B_{jm}^s) \end{pmatrix} \\
&\qquad\qquad + V_{\ell m}^{(ij)+}(\mathcal{H}^{(2)}) \begin{pmatrix} (B_{i\ell}^c A_{jm}^c + B_{i\ell}^s A_{jm}^s) \\ (B_{i\ell}^s A_{jm}^c - B_{i\ell}^c A_{jm}^s) \end{pmatrix} \Big]_{m=\ell+n} \\
&\qquad + \sum_{\ell=1}^{n-1} \Big[W_{\ell m}^{(ij)+}(\mathcal{F}^{(2)}) \begin{pmatrix} (B_{i\ell}^c B_{jm}^c - B_{i\ell}^s B_{jm}^s) \\ (B_{i\ell}^s B_{jm}^c + B_{i\ell}^c B_{jm}^s) \end{pmatrix} \\
&\qquad\qquad + W_{\ell m}^{(ij)+}(\mathcal{G}^{(2)}) \begin{pmatrix} (A_{i\ell}^c B_{jm}^c - A_{i\ell}^s B_{jm}^s) \\ (A_{i\ell}^s B_{jm}^c + A_{i\ell}^c B_{jm}^s) \end{pmatrix} \\
&\qquad\qquad + W_{\ell m}^{(ij)+}(\mathcal{H}^{(2)}) \begin{pmatrix} (B_{i\ell}^c A_{jm}^c - B_{i\ell}^s A_{jm}^s) \\ (B_{i\ell}^s A_{jm}^c + B_{i\ell}^c A_{jm}^s) \end{pmatrix} \Big]_{m=n-\ell} \Big\}
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
&\int_b^\infty H_n^{(2)}(\kappa_k\rho)\rho d\rho \int_0^{2\pi} Q_{ij}^-(\rho,\theta)\begin{pmatrix}\cos n\theta\\ \sin n\theta\end{pmatrix} d\theta = \\
&\quad \frac{\pi i}{8} b^2 \Big\{ \sum_{\ell=n}^{M} (1+\delta_{\ell n}) \Big[U_{\ell m}^{(ij)-}(\mathcal{F}^{(1)}) \begin{pmatrix}(B_{i\ell}^c B_{jm}^{*c} + B_{i\ell}^s B_{jm}^{*s})\\ (B_{i\ell}^s B_{jm}^{*c} - B_{i\ell}^c B_{jm}^{*s})\end{pmatrix} \\
&\qquad + U_{\ell m}^{(ij)-}(\mathcal{G}^{(1)}) \begin{pmatrix}(A_{i\ell}^c B_{jm}^{*c} + A_{i\ell}^s B_{jm}^{*s})\\ (A_{i\ell}^s B_{jm}^{*c} - A_{i\ell}^c B_{jm}^{*s})\end{pmatrix} \\
&\qquad + U_{\ell m}^{(ij)-}(\mathcal{H}^{(1)}) \begin{pmatrix}(B_{i\ell}^c A_{jm}^{*c} + B_{i\ell}^s A_{jm}^{*s})\\ (B_{i\ell}^s A_{jm}^{*c} - B_{i\ell}^c A_{jm}^{*s})\end{pmatrix} \Big]_{m=\ell-n} \\
&\quad \pm \sum_{\ell=0}^{M-n} (1+\delta_{\ell 0}) \Big[V_{\ell m}^{(ij)-}(\mathcal{F}^{(1)}) \begin{pmatrix}(B_{i\ell}^c B_{jm}^{*c} + B_{i\ell}^s B_{jm}^{*s})\\ (B_{i\ell}^s B_{jm}^{*c} - B_{i\ell}^c B_{jm}^{*s})\end{pmatrix} \\
&\qquad + V_{\ell m}^{(ij)-}(\mathcal{G}^{(1)}) \begin{pmatrix}(A_{i\ell}^c B_{jm}^{*c} + A_{i\ell}^s B_{jm}^{*s})\\ (A_{i\ell}^s B_{jm}^{*c} - A_{i\ell}^c B_{jm}^{*s})\end{pmatrix} \\
&\qquad + V_{\ell m}^{(ij)-}(\mathcal{H}^{(1)}) \begin{pmatrix}(B_{i\ell}^c A_{jm}^{*c} + B_{i\ell}^s A_{jm}^{*s})\\ (B_{i\ell}^s A_{jm}^{*c} - B_{i\ell}^c A_{jm}^{*s})\end{pmatrix} \Big]_{m=\ell+n} \\
&\quad + \sum_{\ell=1}^{n-1} \Big[W_{\ell m}^{(ij)-}(\mathcal{F}^{(1)}) \begin{pmatrix}(B_{i\ell}^c B_{jm}^{*c} - B_{i\ell}^s B_{jm}^{*s})\\ (B_{i\ell}^s B_{jm}^{*c} + B_{i\ell}^c B_{jm}^{*s})\end{pmatrix} \\
&\qquad + W_{\ell m}^{(ij)-}(\mathcal{G}^{(1)}) \begin{pmatrix}(A_{i\ell}^c B_{jm}^{*c} - A_{i\ell}^s B_{jm}^{*s})\\ (A_{i\ell}^s B_{jm}^{*c} + A_{i\ell}^c B_{jm}^{*s})\end{pmatrix} \\
&\qquad + W_{\ell m}^{(ij)-}(\mathcal{H}^{(1)}) \begin{pmatrix}(B_{i\ell}^c A_{jm}^{*c} - B_{i\ell}^s A_{jm}^{*s})\\ (B_{i\ell}^s A_{jm}^{*c} + B_{i\ell}^c A_{jm}^{*s})\end{pmatrix} \Big]_{m=n-\ell} \Big\}
\end{aligned}
\tag{4.19}
$$

Special attention is required when the integers ℓ, m are equal to zero. Since the Bessel and Hankel functions are odd functions of the (integer) order it follows that $\mathcal{F}_{-1,m,n}^{(1,2)} = -\mathcal{F}_{1,m,n}^{(1,2)}$, and similarly for the case $m = \pm 1$ and for the integrals (4.13b,c). Thus it follows from (4.17) that $U_{\ell 0} = W_{\ell 0}$ and $V_{0m} = W_{0m}$. It can then be shown, with regard for the limits of each sum in (4.18-19), and for the factors of two applied to the first term in each of the first two sums, that the contributions from these equations are equivalent to those from (4.14-15) when one or more of the integers ℓ, m, n is equal to zero.

Note also that when $n = 0$ the contributions from the first two sums in (4.18-19) are identical, giving a net contribution of twice that of the first sum in the case of the cosine integrals (upper factors) and zero for the sine integrals (lower factors).

4.4. EVALUATION OF THE RADIAL INTEGRALS

Suitable algorithms are required to evaluate the three families of integrals defined by (4.13). From the asymptotic approximations of the Bessel and Hankel functions it is apparent that, for sufficiently large values of x, the integrands are proportional to $x^{-1/2}\exp(i\sigma x)$, where the factor σ involves sums and differences of the three parameters α, β, γ. These integrals are convergent provided $\sigma \neq 0$. Hereafter it is assumed that the three parameters $\alpha, \beta, \gamma > 0$, and $\sigma \neq 0$. The former assumption is appropriate since these parameters correspond physically to the products of three positive wavenumbers with the partition radius b. The latter assumption will be considered in future work.

Numerical integration along the semi-infinite real x-axis is not practical, due to the very slow convergence and oscillatory feature of the integrand. These difficulties can be overcome simply by adopting appropriate contours of integration in the complex z-plane, depending on the sign of σ. To determine the appropriate contours for each of the integrals (4.13), consideration must be given to the basic definitions $H_\nu^{(1,2)} = J_\nu \pm iY_\nu$, where the Bessel functions J, Y are real on the real axis, and the corresponding asymptotic approximations

$$H_\nu^{(1,2)}(z) \simeq \sqrt{\frac{2}{\pi z}} \exp(\pm i(z - \frac{1}{2}\nu\pi - \frac{\pi}{4})) \tag{4.20}$$

which are valid for $|z| >> 1$ and $|z| >> \nu$. When the order is large it also is necessary to consider the complementary approximations

$$J_\nu(z) \sim z^\nu, \qquad Y_\nu \sim z^{-\nu} \tag{4.21}$$

¿From the basic relations between the Hankel and Bessel functions the integrals (4.13c) can be evaluated as one-half times the sum of the two integrals defined by (4.13a), and a similar approach can be followed with (4.13b). Such a decomposition is essential in some cases for the contour integrals, since for large values of the argument $J_\nu(z)$ includes two components proportional respectively to $\exp(\pm iz)$ which may require different choices of the contour of integration. However when $\ell >> \alpha x$ or $m >> \beta x$ it is clear from (4.21) that severe cancellation errors will result from this approach. This difficulty is avoided by integrating (4.13) along a finite segment of the positive real axis $1 \leq x \leq x_{max}$, before deforming the contour of integration into the complex plane. This procedure is robust provided the parameter x_{max} is sufficiently large to satisfy the inequalities $\ell \leq \alpha x_{max}$ and $m \leq \beta x_{max}$.

The remaining integrals are of the general form

$$\int_{x_{max}}^{\infty} f(x) \mathrm{e}^{i\sigma x} dx \tag{4.22}$$

where $f(x)$ denotes a slowly-varying function of order $x^{-1/2}$, for large x, and in all cases $\sigma = (\pm\alpha \pm \beta - \gamma)$. If $\sigma > 0$ the contour of integration in (4.22) may be deformed in the first quadrant of the complex plane $z = x + iy$, and a convenient choice is the straight line upward parallel to the imaginary axis from $z = x_{max}$ to $z = x_{max} + i\infty$. Thus (4.22) is replaced in this case by the semi-infinite integral

$$\int_{x_{max}}^{\infty} f(x)\mathrm{e}^{i\sigma x}dx = i\exp(i\sigma x_{max})\int_{0}^{\infty} f(x_{max}+iy)\mathrm{e}^{-\sigma y}dy, \quad (\sigma > 0) \tag{4.23}$$

Conversely, if $\sigma < 0$, the contour of integration may be deformed in the fourth quadrant along a similar contour parallel to the negative imaginary axis.

For programming purposes it is convenient to modify these integrals so that in all cases the contour of integration is in the first quadrant. Thus, when $\sigma < 0$, (4.22) is replaced by

$$\begin{aligned}\int_{x_{max}}^{\infty} f(x)\mathrm{e}^{i\sigma x}dx = \Big(\int_{x_{max}}^{\infty} f^*(x)\mathrm{e}^{-i\sigma x}dx\Big)^* = \\ -i\int_{0}^{\infty}\Big(f^*(x_{max}+iy)\mathrm{e}^{-i\sigma x+\sigma y}\Big)^* dy \quad (\sigma < 0)\end{aligned} \tag{4.24}$$

Note that the function $f(z)$ is defined, apart from a multiplicative exponential factor, by the integrands of (4.13), and the conjugate function $f^*(z)$ therefore involves the same integrands except that the Hankel functions of first- and second-kind are reversed.

In both the finite integral along the real axis and the semi-infinite integrals (4.23-4) numerical integration is practical using adaptive Romberg quadratures to ensure a specified tolerance of accuracy. It is computationally efficient to evaluate all three integrals (4.13a,b,c) together, for all required combinations of the orders (ℓ, m, n), with the same integration algorithm and different integrands. The domain of integration is subdivided, based on the maximum value of $|\sigma|$, so that large values of this parameter do not retard convergence. The parameter x_{max} is set equal to the largest value of the ratios ℓ/α or m/β, except that if the largest ratio is less than 1.1, x_{max} is set equal to one and the integral along the real axis is skipped. For the semi-infinite integrals in y, the length of the first segment is set equal to four divided by the maximum value of the exponential factor σ. The length of each subsequent segment is increased in a linear manner, proportional to the number of the segment, to take advantage of the exponentially diminishing magnitude of the integrand in that segment. Two convergence tests are required, first within each finite segment to determine that the order of the Romberg quadratures is sufficient, and secondly after the (converged) integral in each segment is added to the total integral to determine when the range of integration can be truncated. Typically a tolerance of 10^{-8} is used, with absolute accuracy specified if the integral is smaller than one in absolute value, and relative accuracy in the converse case. In the program this procedure is followed for the complete set of required integrals, of all combinations of the orders ℓ, m, n required in (4.18-19) and for all three of the integrals defined in (4.13). Convergence tests are applied to all members of this set simultaneously, and the integration of the entire set continues until the convergence test is satisfied for the entire set. This precludes the possibility of false convergence for one particular integrand, and permits the simultaneous recursive evaluation of the required Bessel and Hankel functions of all orders.

Effective subroutines for the Bessel and Hankel functions are required, for both the integral along the real axis where the arguments are all real, and also in the integrals (4.23-4) where the arguments are in the first quadrant. These will be described separately below.

For the functions with real argument the functions $J_\nu(x)$ and $Y_\nu(x)$ are evaluated following the general methodology described in Numerical Recipes, §6.4, and the Hankel functions are evaluated from the relations $J \pm iY$. For $J_\nu(x)$ the backward recursion algorithm of Miller (cf. Abramowitz & Stegun, pp. 385-6) is used with the starting value $\nu = M$ chosen to be the smallest even integer M which satisfies the inequalities $M > 1.36|x| + 24$ and $M > N + 12$, where N is the maximum order required. Normalization is based on the Neumann series with unit value (Abramowitz & Stegun, equation 9.1.46). Numerical experiments indicate that the accuracy of the results is at least 14 decimals, with relative accuracy preserved for $x << \nu$. Chebyshev expansions are used to evaluate the functions Y_0, Y_1 and the functions Y_ν are evaluated for all required higher orders by forward recursion.

A similar backward recursion algorithm is used to evaluate the functions $J_\nu(z)$ with complex argument z in the first quadrant, using the same starting value M defined above except that $|x|$ is replaced by $|z|$. In this case it is appropriate to use as the normalization relations the Neumann series for $\cos, \sin z$ (Abramowitz & Stegun, equations 9.1.47-8), selecting whichever of these functions has the largest absolute value to avoid instability near the zeros on the real axis. To avoid the possibility of overflow, the subroutine evaluates the normalized functions $\exp(iz)J_\nu(z)$ and applies an appropriate exponential factor to the final integrand.

For the Hankel functions of the first kind, with complex argument z in the first quadrant, forward recursion is used based on the starting values of order zero and one. The evaluation of these is based on two complementary algorithms, suitable for small and large values of $|z|$. For small $|z|$ ascending series are summed to evaluate $J_{0,1}$ and $Y_{0,1}$, (Abramowitz & Stegun equations 9.1.10-11), and these are then combined to form the corresponding Hankel functions. These ascending series are used in the quadrilateral domain of the plane $z = x + iy$ such that $x \geq 0$, $0 \leq y \leq 3$ and $x + y \leq 8$. The series are truncated with a total of 22 terms. For larger values of z outside the quadrilateral domain, rational fraction approximations of the form described by Luke (Table 66) are effective. To obtain sufficient accuracy in the domain where $|z|$ is large, the coefficients in Luke's Tables 66.1 and 66.6 have been extended to $n = 10$, giving rational-fraction approximations as the ratios of tenth-degree polynomials. Comparison of the two complementary algorithms along the common boundary of the partition indicates that the minimum relative accuracy of this subroutine is 14 decimals. The rational-fraction approximations are not effective for the Hankel functions of the second kind, but the latter can be evaluated directly from the relation $H_\nu^{(2)} = 2J_\nu - H_\nu^{(1)}$, using the values of J_ν obtained from backward recursion. The exponential normalization factors $\exp(\mp iz)$ are applied to $H_\nu^{(1,2)}$, respectively, and corrected in the final evaluation of the integrand.

For checking purposes it is useful to derive recursion formulae for the integrals (4.13). For this purpose we consider the more general integrals

$$\mathcal{F}_{\ell mn}^{(\mu)} = \int_1^\infty H_\ell(\alpha x) H_m(\beta x) H_n(\gamma x) x^\mu dx \tag{4.25}$$

where H_ν denotes any of the three functions J_ν, $H_\nu^{(1)}$, or $H_\nu^{(2)}$. These integrals are defined in the ordinary sense for $\mu < 3/2$ (assuming the sum of the wavenumbers is nonzero), and will be applied ultimately with $\mu = 1$. Integrating the derivative of the integrand and employing the formula $H'_\nu = H_{\nu-1} - (\nu/z)H_\nu$ to differentiate the integrand in accordance with the chain rule, it follows that

$$-H_\ell(\alpha)H_m(\beta)H_n(\gamma) = \alpha\mathcal{F}^{(\mu)}_{\ell-1,m,n} + \beta\mathcal{F}^{(\mu)}_{\ell,m-1,n} + \gamma\mathcal{F}^{(\mu)}_{\ell,m,n-1} - (\ell+m+n-\mu)\mathcal{F}^{(\mu-1)}_{\ell,m,n} \quad (4.26)$$

Using the recursion $2\nu H_\nu/z = H_{\nu-1} + H_{\nu+1}$ to evaluate the last term in (4.13), with the index ν replaced successively by ℓ, m, n, three relations are obtained as follows:

$$\mathcal{F}^{(\mu-1)}_{\ell,m,n} = \frac{\alpha}{2\ell}[\mathcal{F}^{(\mu)}_{\ell-1,m,n} + \mathcal{F}^{(\mu)}_{\ell+1,m,n}] = \frac{\beta}{2m}[\mathcal{F}^{(\mu)}_{\ell,m-1,n} + \mathcal{F}^{(\mu)}_{\ell,m+1,n}] = \frac{\gamma}{2n}[\mathcal{F}^{(\mu)}_{\ell,m,n-1} + \mathcal{F}^{(\mu)}_{\ell,m,n+1}] \quad (4.27)$$

Combining each of these alternative relations with (4.26) and setting $\mu = 1$ gives the desired relations

$$\mathcal{F}_{\ell+1,m,n} = \frac{2\ell}{\alpha(\ell+m+n-1)}\Big[\alpha\mathcal{F}_{\ell-1,m,n} + \beta\mathcal{F}_{\ell,m-1,n} + \gamma\mathcal{F}_{\ell,m,n-1} + H_\ell(\alpha)H_m(\beta)H_n(\gamma)\Big] - \mathcal{F}_{\ell-1,m,n} \quad (4.28a)$$

$$\mathcal{F}_{\ell,m+1,n} = \frac{2m}{\beta(\ell+m+n-1)}\Big[\alpha\mathcal{F}_{\ell-1,m,n} + \beta\mathcal{F}_{\ell,m-1,n} + \gamma\mathcal{F}_{\ell,m,n-1} + H_\ell(\alpha)H_m(\beta)H_n(\gamma)\Big] - \mathcal{F}_{\ell,m-1,n} \quad (4.28b)$$

$$\mathcal{F}_{\ell,m,n+1} = \frac{2n}{\gamma(\ell+m+n-1)}\Big[\alpha\mathcal{F}_{\ell-1,m,n} + \beta\mathcal{F}_{\ell,m-1,n} + \gamma\mathcal{F}_{\ell,m,n-1} + H_\ell(\alpha)H_m(\beta)H_n(\gamma)\Big] - \mathcal{F}_{\ell,m,n-1} \quad (4.28c)$$

The relations used here for differentiation and recursion of the Hankel functions apply to any linear combination of Bessel functions of the first and second kind, including the Hankel functions of the first and second kind. Thus the same generality applies to the relations (4.28). In particular, one of the Hankel functions may be replaced by the Bessel function J_ν, as is required in (4.13b-c). The only requirements here are that the starting values for the recursion are evaluated for the appropriate combination of Bessel and Hankel functions, and that the triple product of Hankel functions $H_\ell(\alpha)H_m(\beta)H_n(\gamma)$ which appears in (4.28) is likewise replaced by the corresponding product of the appropriate Bessel and Hankel functions, i.e. the integrand of (4.13a,b,c) evaluated at the lower limit of integration.

Collectively the three recursion relations (4.28) can be used to evaluate the integrals $\mathcal{F}, \mathcal{G}, \mathcal{H}$ for any combination of the three orders ℓ, m, n, provided suitable starting values are available. To proceed in this manner throughout a three-dimensional cube $0 \leq (\ell, m, n) \leq M$ it is sufficient to evaluate the eleven starting values where the three integers have the values 000, 100, 010, 001, 011, 101, 110, 111, 200, 020, 002. However this procedure has two important disadvantages. First, to evaluate only the required 'two-dimensional' combinations of ℓ, m, n in (4.18) and (4.19) it is not efficient to evaluate the 'three-dimensional' combinations as is required in the recursion relations. More importantly, for large values of α, β, γ the recursion relations are unstable with substantial cancellation error occurring over several successive values of the integers. (Since forward recursion is stable for the Hankel functions it was originally thought that this problem would not be serious. However the inhomogeneous terms in (4.28) appear to be important in this context, and numerical tests indicate that there is a substantial loss of accuracy in the use of (4.28) for $\ell, m, n > O(10)$ unless the parameters $\alpha, \beta, \gamma = O(1)$.)

The recursion relations are nevertheless useful to check the subroutines used to evaluate (4.13). Extensive tests have been made in this manner, throughout the two-dimensional space where the three integrals (4.13) are required in (4.18-19) up to the maximum order $M = 32$; with a specified Romberg tolerance (typically 10^{-8}) the differences between the left- and right-hand sides of the recursion relations (4.28) are consistently smaller than this tolerance.

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