A simplified derivation of the ordinary differential equations for the free-surface Green functions

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Abstract

Ordinary differential equations (ODEs) are derived for the free-surface Green functions and their gradients, in a fluid of infinite depth. The cases of harmonic time-dependence in the frequency domain and impulsive motion in the time domain are considered separately. These ODEs were derived originally by Clément, starting with fourth-oder equations in the time domain and using Fourier transforms to derive a second-order ODE in the frequency domain. In the present work a simpler procedure is followed independently in each domain, transforming the governing Laplace equation to an ordinary differential equation. The results are consistent with the ODEs derived using Clément's method.

1 Introduction

The free-surface Green function is frequently used to perform computations of wave-body interactions. This function is equivalent to the velocity potential of a submerged source. It can be represented by the sum of a Rankine source and the remaining part which depends on the boundary conditions at the free surface and bottom. The Rankine component is simply the inverse of the distance between the field point $\mathbf{x}$ and the source point $\boldsymbol{\xi}$ in three dimensions. It is excluded from the following discussion, which pertains solely to the additional component required to satisfy the boundary conditions (as well as the radiation condition in the frequency domain or initial conditions in the time domain).

If the boundary conditions on the free surface are linearized and the fluid depth is constant, or sufficiently large to be considered infinite, the Green function can be expressed as an integral over the relevant wavenumber space. Comprehensive summaries are given by Wehausen & Laitone [1] (Section 13) and Linton & McIver [2] (Appendix B). In the cases considered here the fluid depth is assumed infinite. This is a relatively simple case, but one of frequent use in the radiation/diffraction analysis of offshore structures. This type of analysis is usually performed in the frequency domain, using the Green function with harmonic time-dependence. Various algorithms have been developed for computations including series expansions and multi-dimensional polynomial approximations, as well as direct numerical integration.

Clément [3] has shown that the impulsive Green function, with strength proportional to a delta function in the time domain, satisfies a fourth-order ordinary differential equation (ODE) in time. Similar ODEs are also derived in [3] for the gradient of this function. Subsequently in [4] Clément showed by Fourier analysis that the time-harmonic Green function satisfies a second-order ODE in the frequency. These are radical developments, which may be useful for computations. Applications of this approach in the frequency domain are discussed by Shen *et al* [5] and Xie *et al* [6,7]. Shen *et al* [5] include ODEs for the gradient of the time-harmonic Green function, which are shown without derivations but stated to be based on the results in [3] following a similar procedure as in [4].

In the present work a simpler and more direct derivation is presented, first for the time-harmonic Green function and separately for the impulsive function. These derivations follow from the fact that the Green function is a solution of the Laplace equation. The Green function for harmonic time dependence and infinite depth is a function of only two nondimensional coordinates, denoted here by R and Z, corresponding to the horizontal and vertical positions of the field point relative to the image of the source above the free surface and normalized by the wavenumber K. The derivatives with respect to Z have relatively simple forms, which follow from the free-surface boundary condition. The derivatives with respect to R can be related to total derivatives with respect to K. Thus the ODEs can be derived directly from the Laplace equation, involving only derivatives of first and second order with respect to the wavenumber. Since $K = \omega^2/g$, where ω is the frequency and g the gravitational acceleration, it is straightforward to derive corresponding ODEs with respect to the frequency. The approach for the impulsive Green function is similar.

The ODEs for the time-harmonic Green function are derived in Section 2, and for its gradient in Section 3. In Section 4 these results are shown to be consistent with the ODEs in [4-7]. The impulsive Green function and its gradient are considered in Section 5. The results are discussed in the concluding Section 6.

2 ODEs for the time-harmonic Green function

The motion is harmonic with frequency ω and the time-dependent factor $\mathrm{e}^{\mathrm{i}\omega t}$ is assumed. The corresponding wavenumber is $K = \omega^2/g$. A Cartesian coordinate system $\mathbf{x} = (x_1, x_2, x_3)$ is defined with $x_3 = 0$ the plane of the free surface and $x_3 < 0$ the domain of the fluid. The source is at $\mathbf{x} = \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$, with $\xi_3 \leq 0$. The image point of the source above the free surface is $\boldsymbol{\xi}' = (\xi_1, \xi_2, -\xi_3)$. Cylindrical and spherical coordinates are defined as follows:

$$r = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}, \quad z = x_3 + \xi_3, \quad s = |\mathbf{x} - \boldsymbol{\xi}'| = \sqrt{r^2 + z^2}. \tag{1}$$

Thus (r, z) is a cylindrical coordinate system with radius r and vertical coordinate z, with the origin at the image of the source point and z positive upward. The spherical radius s is the distance between the field point and image source point, with horizontal component r and vertical component z. Nondimensional coordinates are defined by

$$R = Kr, \quad Z = Kz, \quad S = Ks. \tag{2}$$

With these definitions the Green function can be defined in the form (cf. [1], equation 13.17, or [2], Appendix B.3)

$$G(\mathbf{x};\boldsymbol{\xi}) = \frac{1}{|\mathbf{x}-\boldsymbol{\xi}|} + \frac{1}{|\mathbf{x}-\boldsymbol{\xi}`|} + KF(R,Z), \tag{3}$$

where

$$F(R,Z) = 2\int_0^\infty \frac{dk}{k-1} \mathrm{e}^{kZ} J_0(kR) \tag{4a}$$

$$= -\pi \mathrm{e}^{Z}\left(\mathbf{H}_0(R) + Y_0(R) + 2\mathrm{i}J_0(R)\right) - 2\int_0^{|Z|} \mathrm{e}^{\zeta+Z}\left(R^2+\zeta^2\right)^{-\frac{1}{2}} d\zeta. \tag{4b}$$

Here J_0 and Y_0 are the Bessel functions of the first and second kind and $\mathbf{H}_0$ is the Struve function. The contour of integration in (4a) is defined to pass above the pole $k = 1$ in the complex $k-$plane, to satisfy the radiation condition.

Since the Green function satisfies the Laplace equation, it follows that

$$\nabla^2 F = F_{rr} + (1/r)F_r + F_{zz} = K^2\Big(F_{RR} + (1/R)F_R + F_{ZZ}\Big) = 0. \tag{5}$$

The following relations are used to evaluate the partial derivatives in (5):

$$\begin{aligned} F_Z &= F + 2/S && (6)\\ F_{ZZ} &= F_Z - 2Z/S^3 = F - 2Z/S^3 + 2/S && (7)\\ F_{RZ} &= F_R - 2R/S^3 && (8)\\ KF_K &= RF_R + ZF_Z && (9)\\ K^2F_{KK} &= R^2F_{RR} + 2RZF_{RZ} + Z^2F_{ZZ} && \\ &= R^2F_{RR} + 2RZF_R - 4R^2Z/S^3 + Z^2F_{ZZ}. && (10) \end{aligned}$$

Note that subscripted coordinates (R,Z) denote the partial derivatives, whereas subscripts of the wavenumber K denote the total derivatives with respect to this parameter. (6) can be derived most easily from (4b), or alternatively from (4a) using the integral relation for $1/S$ (cf [1], equation 13,12). (7-8) follow directly by differentiation of (6). (9-10) simply express the total derivatives in accordance with the definitions of R and Z and the fact that F depends only on these two nondimensional coordinates.

Using (9-10) first to replace F_R and F_{RR} in (5),

$$\begin{aligned} (R^2/K^2)\nabla^2 F &= K^2F_{KK} + (1-2Z)\,KF_K + 4R^2Z/S^3 \\ &\qquad -Z\,(1-2Z)\,F_Z + \left(R^2 - Z^2\right)F_{ZZ} = 0. \end{aligned} \tag{11}$$

Using (6-7) to replace F_Z and F_{ZZ} gives the ODE

$$\begin{aligned} K^2F_{KK} &+ (1-2Z)\,KF_K + \left(S^2 - Z\right)F = \\ &-4R^2Z/S^3 + 2Z\,(1-2Z)\,/S + 2Z\left(R^2-Z^2\right)/S^3 - 2\left(R^2-Z^2\right)/S. \end{aligned} \tag{12}$$

After simplifying the right-hand side the final form for the ODE is given by

$$K^2F_{KK} + (1-2Z)\,KF_K + \left(S^2 - Z\right)F = -2S. \tag{13}$$

Alternatively, in terms of the frequency $\omega = \sqrt{gK}$,

$$Kd/dK = \tfrac{1}{2}\omega d/d\omega, \quad K^2d^2/dK^2 = \tfrac{1}{4}\omega^2 d^2/d\omega^2 - \tfrac{1}{4}\omega d/d\omega.$$

Thus

$$\tfrac{1}{4}\omega^2 F_{\omega\omega} + (\tfrac{1}{4} - Z)\omega F_\omega + \left(S^2 - Z\right)F = -2S. \tag{14}$$

3 Gradient of the Green function

Similar results can be derived for the derivatives $U = F_r$ and $V = F_z$, which usually are required for computations. For $U = KF_R$ the relations analogous to (6-10) are

$$\begin{aligned} U_Z &= U - 2KR/S^3 && (15)\\ U_{ZZ} &= U_Z + 6KRZ/S^5 && (16)\\ U_{RZ} &= U_R + 6KR^2/S^5 - 2K/S^3 && (17)\\ KU_K &= RU_R + ZU_Z + U && (18)\\ K^2U_{KK} &= R^2U_{RR} + 2RZU_{RZ} + Z^2U_{ZZ} + 2RU_R + 2ZU_Z && (19) \end{aligned}$$

If (5) is differentiated with respect to R it follows that

$$K^2\Big(F_{RRR} + F_{RR}/R - F_R/R^2 + F_{RZZ}\Big) = K\Big(U_{RR} + U_R/R - U/R^2 + U_{ZZ}\Big) = 0 \quad (20)$$

Multiplying (20) by R^2/K and using (15-19) gives the ODE

$$K^2U_{KK} - (1+2Z)KU_K + (S^2+Z)U = 2KR/S \quad (21)$$

or

$$\tfrac{1}{4}\omega^2 U_{\omega\omega} - (Z+\tfrac{3}{4})\omega U_\omega + \Big(S^2+Z\Big)U = 2KR/S. \quad (22)$$

The ODE for V can be derived directly from (13), since

$$V = KF_Z = KF + 2K/S. \quad (23)$$

After substituting $F = V/K - 2/S$ in (13) and multiplying by K,

$$K^2V_{KK} - (1+2Z)\,KV_K + \Big(S^2+Z+1\Big)V = 2K\,(1+Z)\,/S \quad (24)$$

or

$$\tfrac{1}{4}\omega^2 V_{\omega\omega} - (Z+\tfrac{3}{4})\omega V_\omega + \Big(S^2+Z+1\Big)V = 2K\,(1+Z)\,/S. \quad (25)$$

4 ODEs for Clément's form of the Green function

In [4] the Green function is represented in the alternative form

$$G(\mathbf{x};\boldsymbol{\xi}) = \frac{1}{|\mathbf{x}-\boldsymbol{\xi}|} - \frac{1}{|\mathbf{x}-\boldsymbol{\xi}'|} + \tilde{F}. \quad (26)$$

Comparing (26) with (3) it follows that $\tilde{F} = KF + 2K/S$. Since this is the same relation as (23), it follows that $\tilde{F}$ satisfies the ODEs (24-25). Except for the normalization of the coordinates (25) is identical to the ODE derived by Clément ([4], equation 11).

The corresponding components of the gradient are

$$\tilde{U} = \tilde{F}_r = KU - 2K^2R/S^3,$$

$$\tilde{V} = \tilde{F}_z = KV - 2KZ/S^3.$$

Using these relations with (21-22) and (24-25) gives the results

$$K^2\tilde{U}_{KK} - (2Z+3)\,K\tilde{U}_K + \left(S^2+3Z+3\right)\tilde{U} = -6(1+Z)K^2R/S^3 \tag{27}$$

$$K^2\tilde{V}_{KK} - (2Z+3)\,K\tilde{V}_K + \left(S^2+3Z+4\right)\tilde{V} = (2R^2-4Z^2-8Z)K^2/S^3 \tag{28}$$

and

$$\tfrac{1}{4}\omega^2\tilde{U}_{\omega\omega} - (Z+\tfrac{7}{4})\omega\tilde{U}_\omega + \left(S^2+3Z+3\right)\tilde{U} = -6(1+Z)K^2R/S^3 \tag{29}$$

$$\tfrac{1}{4}\omega^2\tilde{V}_{\omega\omega} - (Z+\tfrac{7}{4})\omega\tilde{V}_\omega + \left(S^2+3Z+4\right)\tilde{V} = (2R^2-4Z^2-8Z)K^2/S^3. \tag{30}$$

Except for the normalization of the coordinates (28) and (30) are the same as the results given by Shen *et al* ([3], equations 6-7).

5 ODEs for the time-domain Green function

Using the dimensional coordinates (1), the Green function for impulsive motion in the time domain is defined by (cf. [1], equation 13.49)

$$G(\mathbf{x};\boldsymbol{\xi},t) = \frac{\delta(t)}{|\mathbf{x}-\boldsymbol{\xi}|} - \frac{\delta(t)}{|\mathbf{x}-\boldsymbol{\xi}'|} + H(t)f(r,z,t). \tag{31}$$

Here t denotes time, $\delta(t)$ is the Dirac delta function, $H(t)$ the Heaviside step function, and

$$f(r,z,t) = 2\int_0^\infty \sqrt{gk}\sin\left(t\sqrt{gk}\right)\mathrm{e}^{kz}J_0(kr)dk \tag{32a}$$

$$= 2r^{-3/2}\int_0^\infty \sqrt{g\kappa}\sin\left(t\sqrt{g\kappa/r}\right)\mathrm{e}^{\kappa z/r}J_0(\kappa)d\kappa. \tag{32b}$$

The variable of integration $\kappa = kr$ is used in (32b) to avoid differentiation of the Bessel function. This integral depends only on the two nondimensional parameters gt^2/r and z/r, but it is simpler in the following analysis to continue using the dimensional coordinates. From (32a) it follows that

$$f_{tt} = -gf_z, \quad f_{tttt} = g^2 f_{zz}, \tag{33}$$

and, from (32b),

$$rf_r = -\tfrac{3}{2}f - zf_z - \tfrac{1}{2}tf_t = -\tfrac{3}{2}f + (z/g)f_{tt} - \tfrac{1}{2}tf_t. \tag{34}$$

Differentiation of (34) gives the relation

$$\begin{aligned} r^2 f_{rr} + rf_r &= -\tfrac{3}{2}rf_r + (rz/g)f_{rtt} - \tfrac{1}{2}rtf_{rt} \\ &= \tfrac{9}{4}f + \tfrac{7}{4}tf_t + \left[\tfrac{1}{4}t^2 - 4(z/g)\right]f_{tt} - (tz/g)f_{ttt} + (z/g)^2 f_{tttt}. \end{aligned} \tag{35}$$

Using (33) and (35) in the Laplace equation (5) gives the result

$$r^2\nabla^2 f = (s/g)^2 f_{tttt} - (tz/g)f_{ttt} + \left[\tfrac{1}{4}t^2 - 4(z/g)\right]f_{tt} + \tfrac{7}{4}tf_t + \tfrac{9}{4}f = 0. \tag{36}$$

This is the same ODE derived by Clément [3], except for the factor g which is omitted in the normalization used in [3].

The components of the gradient $\mathbf{w} = (f_r, f_z)$ are

$$\mathbf{w} = \begin{pmatrix} u \\ v \end{pmatrix} = 2\int_0^\infty \sqrt{gk}\sin\left(t\sqrt{gk}\right) \mathrm{e}^{kz} \begin{pmatrix} -J_1(kr) \\ J_0(kr) \end{pmatrix} kdk \tag{37a}$$

$$= -2r^{-5/2}\int_0^\infty \sqrt{g\kappa}\sin\left(t\sqrt{g\kappa/r}\right) \mathrm{e}^{\kappa z/r} \begin{pmatrix} -J_1(\kappa) \\ J_0(\kappa) \end{pmatrix} \kappa d\kappa. \tag{37b}$$

The following differential relations are derived from (37a) and (37b) in place of (33) and (35):

$$\mathbf{w}_{tt} = -g\mathbf{w}_z, \quad \mathbf{w}_{tttt} = g^2\mathbf{w}_{zz}, \tag{38}$$

$$\begin{aligned} r^2\mathbf{w}_{rr} + r\mathbf{w}_r &= -\tfrac{5}{2}r\mathbf{w}_r + (rz/g)\mathbf{w}_{rtt} - \tfrac{1}{2}rt\mathbf{w}_{rt} \\ &= \tfrac{25}{4}\mathbf{w} + \tfrac{11}{4}t\mathbf{w}_t + \left[\tfrac{1}{4}t^2 - 6(z/g)\right]\mathbf{w}_{tt} - (tz/g)\mathbf{w}_{ttt} + (z/g)^2\mathbf{w}_{tttt}. \end{aligned} \tag{39}$$

Thus u and v both satisfy the same differential relations (38-39). Recalling the extra term in (20) which only affects the equation for u, it follows from the Laplace equation that

$$(s/g)^2u_{tttt} - (tz/g)u_{ttt} + \left[\tfrac{1}{4}t^2 - 6(z/g)\right]u_{tt} + \tfrac{11}{4}tu_t + \tfrac{21}{4}u = 0. \tag{40}$$

and

$$(s/g)^2v_{tttt} - (tz/g)v_{ttt} + \left[\tfrac{1}{4}t^2 - 6(z/g)\right]v_{tt} + \tfrac{11}{4}tv_t + \tfrac{25}{4}v = 0. \tag{41}$$

These ODEs are consistent with the corresponding equations (6.5) and (6.8) in [3]. The procedure described above explains the similarity of the two equations, which was noted in [3]. The same similarity is evident in the frequency domain, comparing the left-hand sides of (21) and (24) or (22) and (25).

6 Discussion

The principal contribution of this work is a simple and more transparent derivation of the ODEs for the free-surface Green functions, which were derived originally by Clément [3,4] and extended for the gradient in the frequency domain by Shen *et al* [5]. In those works the analysis was performed first in the time domain, leading to fourth-order ODEs with respect to time [3], and then transformed to the frequency domain using Fourier analysis. Here a more direct approach is followed independently for the frequency and time domains.

In the frequency domain second-order ODEs with respect to the wavenumber are derived from the Laplace equation, using differential relations for the vertical and radial derivatives. The corresponding ODEs follow with respect to the frequency. The Green function used for the analysis in Sections 2 and 3 assumes a positive image source above the free surface, as in (3). In [4-7] a negative image source is used, as in (26), and the factor K in the third term of (3) is included in the function $\tilde{F}$. These differences are accounted for in Section 4, to confirm that the ODEs derived here are equivalent to those in [4-7].

The same approach is followed in Section 5 for the impulsive Green function in the time domain, resulting in the fourth-order ODEs derived originally by Clément [3]. The differential relations are more complicated compared to the frequency domain, due to the second time-derivative in the equation for the vertical derivative. This is the principal reason that the ODEs are of order four, compared to second-order in the frequency domain.

Like the previous work in [3-7] our analysis is restricted to the case of infinite depth. This is an important restriction in many practical cases. The present derivations rely on the properties that the Green functions depend on only two nondimensional parameters, and that the vertical derivative can be evaluated from the function itself in the frequency domain, or from the second time-derivative in the time domain. These relations for the vertical derivative correspond to the boundary conditions on the free surface, but in these cases they are valid throughout the fluid domain. In a fluid of finite depth the Green function includes an additional parameter proportional to the depth, and the vertical dependence is more complicated. Thus it is unlikely that our approach can be extended to a fluid of finite depth. This may explain the absence of any similar extension based on the approach used in [3-7]. Also, as in [3], the present derivations rely on the classical solutions for the Green functions, representated by the integrals in (4) and (32), to derive the differential relations (6-10) and (33-35). This precludes the possibility of deriving the ODEs directly from the original boundary-value problems.

After completing this work the author was informed by one of the reviewers that the thesis by Xie [8] includes a similar derivation in the time domain. In that work the ODE is derived from the Laplace equation and relations equivalent to (33), without explicitly using (34). Instead a pair of 'natural variables' (μ, τ) are used to replace the coordinates (r, z) and time t; transformation of the required derivatives leads to a fourth-order ODE with respect to the variable τ, which is equivalent to (36).

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